Eq. (39) is rather curious and we have been unable to attach any particular significance to it.[38,39]

The differences between spinless and spin-$\frac{1}{2}$ constituents are impressive. In particular, while in the spinless case binding by a regular potential meant superconvergence, it is not so for the spin-$\frac{1}{2}$ case. An indication of superconvergence or of $(1/q^2)^2$ behavior of the pion form factor, would imply that either the interaction is strongly regularized by some mechanism (e.g., bootstrap), or that the high-energy model we used is wrong.

[38] We do not know whether the possible existence of $J=0$ singularities in the J plane (Ref. 39) for more sophisticated models can invalidate the composite-particle interpretation of the bound-state solution studied above.

[39] See, e.g., S. Mandelstam and L. L. Wang, Phys. Rev. **160**, 1490 (1967), and references therein.

ACKNOWLEDGMENTS

The author wishes to thank Professor L. A. Radicati for constant encouragement and Professor P. Menotti for useful discussions and suggestions.

PHYSICAL REVIEW VOLUME 176, NUMBER 5 25 DECEMBER 1968

Bootstrap of Meson Trajectories from Superconvergence

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(Received 14 June 1968)

In this paper we study the reactions $\pi\pi\to\pi\omega(1^-)$, $\pi\pi\to\pi A_2(2^+)$, and $\pi\pi\to\pi\omega_3(3^-)$ as a bootstrap system for natural-parity trajectories. We start from the solution of our previous work that gave, among other results, expressions for the trajectory and residue functions as well as mass formulas, in agreement with experiment. Here we study in detail the sum rules as a function of momentum transfer t. We find a set of residue functions $\beta(t)$ that are self-consistent and such that the Regge and resonance sides of the equations are almost equal in a large region of t. We study also a step-by-step approximation that, at each stage, enlarges the region where the equations are valid. We find, however, that the leading Regge trajectories, even if infinitely rising, cannot bootstrap themselves. We outline two possible (not incompatible) ways of implementing the bootstrap. The first way demands an optimized choice of the cutoff parameter and considers the whole family of reactions $\pi\pi\to\pi X_J$ (X_J being a normal-parity state of spin J). Our results for $J\leq3$ show that this is a definite possibility. The second way is to consider a whole family (parent and daughters) as participating in the bootstrap. We find this possibility also attractive, and as a consequence we find that daughters must be parallel to the parent, for linear trajectories. The properties of our parametrization are also discussed—in particular, the Khuri paradox and the coupling of high-spin resonances to the system. We also compare our results with experiment whenever possible. Our A_2 trajectory, for instance, follows the Gell-Mann mechanism, and the exponential t dependence of our residue functions is perfectly consistent with the one found in recent phenomenological fits to inelastic reactions.

1. INTRODUCTION

It seems that a very promising attempt in elementary-particle theory today can be found in blending the general principles of S-matrix theory, embodied in analyticity, crossing, and unitarity, with the dynamical elements contained in Regge-pole theory. The resulting scheme will, it is hoped, put strong enough restrictions on the scattering amplitudes that the Regge trajectories and their residue functions will be uniquely determined. As a consequence, the spectrum of particles and their couplings will be completely determined and their bootstrap accomplished.

A large number of papers, dealing with the question of analyticity at $t=0$ when the external masses are not equal, have shown that Regge trajectories must appear in families.[1] The Regge functions of the members of the family must obey relations at this point but are undetermined elsewhere. These results have been reached by means of powerful group-theoretical techniques by Toller and collaborators[2] and by Freedman and Wang.[3] A few models have also been solved in some approximation, as the Van Hove model[4] and the Bethe-Salpeter

* Work supported in part by the U. S. Office of Naval Research under Contract No. Nonr-1866(55) and by the U. S. Air Force under Contract No. 49 (638)-1380.

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[1] D. S. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967).

[2] M. Toller, Nuovo Cimento **53A**, 671 (1968), and references therein.

[3] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

[4] R. L. Sugar and J. D. Sullivan, Phys. Rev. **166**, 1515 (1968).

equation,[5] that give information at $t\neq 0$ as well. This requirement of analyticity hints that a complete bootstrap scheme can be achieved only if at least a family of trajectories is taken into account.

A somewhat more phenomenological approach was started by De Alfaro *et al.*[6] when they discovered superconvergence relations and proposed their use in particle physics. Their observation is as follows: If for a process the t-channel helicity flip is sufficiently large, and the internal quantum numbers are such that the leading Regge trajectory is below some given value, the corresponding invariant amplitude A obeys a sum rule of the form

$$\int_{-\infty}^{+\infty} \mathrm{Im}A(\nu,t)d\nu=0\,, \tag{1.1}$$

where $\nu=\frac{1}{4}(s-u)$, and s, t, and u are the Mandelstam variables. Equation (1.1) follows from analyticity and Regge asymptotic behavior and the evaluation of the integral can be performed by means of unitarity. In Ref. 6 and in subsequent papers[7] ImA was approximated by a few low-lying resonances, so that the equations resulted in relations among the parameters of s- and u-channel resonances.

Equation (1.1), as such, is a mathematical consequence of the general assumptions we made at the start. It only becomes physically relevant when some prescription, like the above-mentioned one, is given to calculate the integral: this is called in the literature the saturation problem. Saturation in terms of a finite number of resonances has been shown to lead to difficulties[8] when the equations are required to be exactly satisfied in a certain range of t.

In order to avoid this problem, we have proposed, more recently,[9] a different type of saturation philosophy in which the high-energy part of Eq. (1.1) was explicitly taken into account by use of Regge theory. No interference was allowed (in the imaginary part of the amplitude) between the low-energy resonances and the high-energy part represented by Regge trajectories. Such a philosophy provides for a method of analytic continuation of Eq. (1.1) to values of t where the integral is meaningless. This analytic continuation had been previously derived by subtracting the asymptotic limit from the amplitude and writing Eq. (1.1) for the difference. This last method, first proposed by Igi,[10] was rediscovered by many authors[11] and fully exploited by Dolen, Horn, and Schmid[12] in their "finite-energy sum rules." These new sum rules have the form

$$\int_{\nu_t}^{\bar{\nu}} \nu^n\, \mathrm{Im}A(\nu,t)d\nu=\sum_r \beta_r(t)\frac{\bar{\nu}^{\alpha_r+n+1}}{\alpha_r+n+1}\,, \tag{1.2}$$

where Regge theory demands

$$\mathrm{Im}A(\nu,t) \underset{\nu\to\infty}{\longrightarrow} \textstyle\sum_r \beta_r(t)\nu^{\alpha_r(t)}\,. \tag{1.3}$$

In a few cases, like πN scattering, the experimental data from both low-energy and high-energy fits can be used directly to check the equations. A more interesting application is to use the low-energy data as an input to predict the relevant parameters of high-energy scattering, like the leading Regge trajectory. This has been done in Ref. 12 for πN charge-exchange scattering. The results of the thorough analysis of Dolen, Horn, and Schmidt not only showed an excellent agreement with experiment but also made evident a rather surprising property of the Regge representation, viz., while the physical amplitude begins to differ from the Regge term as soon as important enough resonance appear in the direct channel, the *local average* of the amplitude coincides with the extrapolation of the Regge term up to much lower energies. Since what we need in the sum rule (for low moments) is only the average of the amplitude, this property permits us to cut the integral at rather low energies, thus opening a number of possible applications.

Another way of exploiting the above equations, which is more theoretical and certainly very attractive, is the one we call the "bootstrap" of Regge trajectories. The general idea of this approach is that, for some particular reaction, the amplitude in the resonance region of the direct channel can be obtained by use of crossing as the analytic continuation of the Regge amplitude describing scattering at high energy in the crossed channel. The essence of the problem lies therefore in finding a trajectory and residue function that when introduced as input reproduces itself consistently. Also one must find a parametrization of the scattering amplitude which obeys the constraints of analyticity, unitarity, and crossing symmetry. The simplest model of such a theory is the one based on the narrow-resonance approximation and, consistently, on real rising Regge trajectories. In this frame Eqs. (1.1–2) provide us with a set of algebraic relations in terms of Regge parameters only. This model has also been proposed by Mandelstam[13] and has been first exploited by the authors[14,15] in two

[5] A. R. Swift, Phys. Rev. Letters **18**, 813 (1967).

[6] V. De Alfaro, S. Fubini, G. Furlan, and C. Rossetti, Phys. Letters **21**, 576 (1966).

[7] See for example F. Low, *Berkeley Conference Report* (University of California Press, Berkeley, 1966).

[8] This was noticed by S. Fubini *et al.*, *Report to the Coral Gables Conference 1967* (University of Miami Press, Miami, Fla.,1968).

[9] M. Ademollo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Nuovo Cimento **51A**, 227 (1967).

[10] K. Igi, Phys. Rev. Letters **9**, 76 (1962).

[11] K. Igi and S. Matsuda, Phys. Rev. Letters **18**, 625 (1967); L. D. Soloviev, A. A. Logunov, and A. N. Tavkhelidze, Phys. Letters **24B**, 181 (1967); D. Horn and C. Schmid (unpublished); R. Gatto, Phys. Rev. Letters **18**, 803 (1967).

[12] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. Letters **19**, 402 (1967); Phys. Rev. **166**, 1772 (1968).

[13] S. Mandelstam, Phys. Rev. **166**, 1539 (1968).

[14] M. Ademollo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. Letters **19**, 1402 (1967).

[15] M. Ademollo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Letters **27B**, 99 (1968).

[16] C. Schmid, Phys. Rev. Letters **20**, 628 (1968); P. G. O. Freund, *ibid.* **20**, 235 (1968); C. Schmid and J. Yellin, Phys. Letters **27B**, 19 (1968); M. Bishari, H. R. Rubinstein, A. Schwimmer, and G. Veneziano, following paper, **176**, 1926 (1968).

particularly convenient classes of mesonic reactions and further extended to other mesonic processes by other groups.[16]

It turns out that, for a bootstrap program, meson systems are more advantageous than baryonic ones. Since experimental scattering information, as such, is unavailable, some definite saturation assumption is needed. Nevertheless, the price is worth the advantages because, in the new bootstrap, the meson system (i) can be thought of as essentially decoupled from the baryon system, (ii) possesses often very strong symmetries in the three channels (a situation that can never occur with baryons), and (iii) by appropriate choice of reaction one can suppress the type and number of intermediate states that can contribute. This is because in the reactions chosen there are charge-conjugation and parity-selection rules. The saturation hypotheses of our program are the following: The absorptive part of meson-meson scattering amplitudes can be represented in the following way:

(a) *Low energy*: The contribution of a few resonant states is dominant and the background is negligible.

(b) *High energy*: The amplitude is controlled by a few Regge trajectories.

(c) *Intermediate energy*: We use the property that the extrapolated Regge tail is equal to the averaged amplitude (the so-called new interference model[12]).

Our general investigations will be mainly performed on the reaction $\pi\pi\rightarrow\pi\omega$ (Secs. 2 and 3) which proved to be a very clean and useful theoretical laboratory. We include, in successive stages of approximations, more and more resonances as intermediate states. Accordingly we have to increase the value of the cutoff parameter $\bar{\nu}$.

We first saturate the sum rules by means of the ρ meson alone, finding very good agreement for the t dependence provided that the cutoff $\bar{\nu}$ is chosen midway between the ρ and the first resonance to be left out (ρ_3: $J^P=3^-$). We then include the 3^- state, varying the cutoff consistently, and the agreement is found to be good (and even better) in a much larger region of t without having any free parameter at our disposal.

However, further displacement of the cutoff leads into difficulties; the resonances cannot balance the Regge term anymore.

We conclude that it is impossible for the leading trajectory alone to provide enough strength to the integral by means of its own resonances. This is in agreement with the experimental observation that high-spin resonances are essentially only coupled to neighboring states in angular momentum.[17]

In this paper we investigate two possible and not necessarily incompatible ways out of this problem.

The first consists in looking for a general solution in which an entire family of Regge trajectories is involved (parent plus daughters). We find parallelism of daughter trajectories, at least in the approximation of a linear parent trajectory. The result has been also confirmed[18] by use of the partial-wave projection of the Regge term, recently proposed by Schmid.[19] A positive discovery in this approach is the possibility of getting good agreement in a range of t. This stems from the observation that, while the Regge term is a transcendental function and the contribution of resonances is a polynomial, in each step of approximation the first can be factorized into two functions. One is almost exactly constant in a range of t going from $\alpha(t_{\min})=-J$ to $\alpha(t_{\max})=+J$ (J being the spin of the last resonance included); the other is a polynomial of the same order as that coming from the resonances. Furthermore, the coefficients of the highest powers in t are the same on both sides of the equations and the difference may be filled by lower-spin resonances.

The second approach, proposed in Ref. 15, stemmed from the observation that saturation by resonances seems to be a good approximation at low energies but that no evidence for such saturation exists for intermediate and higher energies.

Consequently, it seems that use of a rather low value of the cutoff parameter of the integral might be the most adequate.

To study the pieces of the trajectory linked to high-spin states, we then suggested raising the external spin as well.[15] We develop this idea in this paper by considering reactions of the form $\pi+\pi\rightarrow\pi+J$, where J goes up to three (see Secs. 2, 4, and 5).

We have then in principle an iteration procedure to compute scattering amplitudes that, with the aforementioned saturation assumptions, transforms into an appealing set of equations providing relations for the parameters of the s, t, and u channels. By use of crossing symmetry they appear related once more and the self-consistency or bootstrap is achieved so that in principle we can predict intercepts and slopes of trajectories as well as ratios of their residue functions. The procedure is both strongly predictive and successful, thus confirming the practicability of the program and the hypothesis underlying it. In particular, it gives strong evidence for the following conclusions:

(a) Our amplitudes are well represented by a few resonances at low energy. One should remember, however, that the examples we considered are reactions where the Pomeranchuk does not contribute and there is no s wave. In fact, there is some evidence that in these two cases resonances may not saturate the sum rules well.[20] (b) Validity of the local-average hypothesis, i.e., that the extrapolated Regge term is equal to an average of the physical amplitude. This hypothesis has

[17] G. F. Chew, 1967 Report to the Solvay Conference (to be published).

[18] H. R. Rubinstein, A. Schwimmer, G. Veneziano, and M. A. Virasoro, Phys. Rev. Letters **21**, 491 (1968).

[19] C. Schmid, Phys. Rev. Letters **20**, 689 (1968).

[20] H. Harari, Phys. Rev. Letters **20**, 1395 (1968).

been recently applied by Chew and Pignotti[21] to the relation between the Deck effect and the A_1 resonance. In this respect we should like to stress that local average does not imply Schwarz sum rules.[22] In fact the local average suggests that the sum rules (2) hold separately for the positive and negative regions of ν. Note, however, that the low positive region of ν may very well include some (below threshold) u-channel resonances. A striking example is the $I=2$, $\pi\rho$ superconvergence sum rule considered in Ref. 22. In that example the local average holds but the Schwarz sum rules are badly violated.

The paper is organized in the following way:

Section 2 studies in detail the reaction $\pi\pi\rightarrow\pi\omega$. Most of the section deals with the first-moment sum rule, where we consider in detail the problems mentioned above. At the end of the section we also discuss Schwarz sum rules[22] and higher-moment sum rules. The former ones are violated, indicating the existence of a fixed pole. However, this fixed pole should be additive in order to keep the dip mechanism effective.[23]

In Sec. 3 we study the problems posed by the t dependence of the residue function $\beta(t)$.

In particular we study (a) the t dependence of our model and how it compares with experiment (we find reasonable agreement for positive and negative t), (b) the mechanisms followed by our trajectories at sense-nonsense points of both signatures (we find the Gell-Mann mechanism), (c) the restrictions found by Khuri on these functions [our $\beta(t)$ obeys them], and (d) the relation of our parametrization to the one chosen by Mandelstam.[13] We use different asymptotic variables and a different choice of the scale factor, implying a different t dependence.

In Sec. 4 we study the reaction $\pi\pi\rightarrow\pi A_2$. This reaction contains new features due to the appearance of two independent helicity amplitudes. We apply the same technique of the previous section and we saturate the system up to and including presumed states of spin 4, finding again a very good solution for a large region of t. We discuss the problem of choices between helicity and invariant amplitudes and show the advantages of the latter choice.

In Sec. 5 we go a step further and consider the hypothetical reaction $\pi\pi\rightarrow\pi\omega_3(3^-)$, assuming that meson trajectories are parallel and that such a particle exists. Results are good and couplings of this state are predicted.

Section 6 deals with a summary of the results and a general outlook.

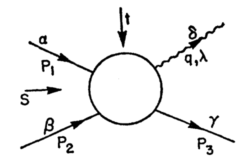
In Appendix A we discuss in detail, in the $SU(3)$ limit, the derivation of the sum rules for $PP\rightarrow PV$ and $PP\rightarrow PT$ presented in the previous work.[14,15]

[21] G. F. Chew and A. Pignotti, Phys. Rev. Letters **20**, 1078 (1968).

[22] J. H. Schwarz, Phys. Rev. **159**, 1269 (1967).

[23] A detailed study of this problem has been performed by R. Roskies (to be published). We thank him for interesting discussions.

FIG. 1. Kinematics of all processes considered in the paper.



Finally, in Appendix B we give the detailed computation of the contribution of resonances up to spin 4 to one of the sum rules that hold for $\pi\pi\rightarrow\pi A_2$. scattering

2. REACTION $\pi\pi\rightarrow\pi\omega$

A most suitable reaction providing for a "bootstrap" of the ρ trajectory is $\pi\pi\rightarrow\pi\omega$. In this section we discuss this example in full detail.

The T matrix for this process is described in terms of a single invariant amplitude $A(\nu,t)$, defined through

$$T^{\alpha\beta\gamma}=\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu\rho\sigma}e_{\mu}^{(\lambda)}P_{1\nu}P_{2\rho}P_{3\sigma}A(\nu,t)\,, \qquad (2.1)$$

where the momenta and isospin indices of the pions are taken as in Fig. 1. A more detailed discussion of this amplitude is given in Appendix A.

The most remarkable property of this reaction is that it selects, among the possible single particles and Regge poles that can be exchanged in all channels, these corresponding to negative signature, $I=1$, $G=+1$, and normal-parity trajectories. The fact that it is an inelastic reaction is unimportant in this bootstrap theory. We first assume that one single trajectory with these properties controls the high-energy scattering. Since the results are that its intercept at $t=0$ is about $\frac{1}{2}$ and the slope is about 1 BeV^{-2}, we identify it with the ρ trajectory. The low-energy absorptive part is assumed in the beginning to be dominated by the exchange in the s and u channels of the particles lying on this same trajectory.

The contribution of the leading Regge pole to the amplitude will be parametrized for high ν and fixed t as

$$A(\nu,t)\underset{\nu\rightarrow\infty}{\longrightarrow}\beta(t)\xi(\alpha)(\nu/\nu_1)^{\alpha(t)-1}\,, \qquad (2.2)$$

where

$$\xi(\alpha)=[1-e^{-i\pi\alpha(t)}]/\sin\pi\alpha(t)\,. \qquad (2.3)$$

We shall also parametrize the residue function $\beta(t)$ in the form

$$\beta(t)=\bar{\beta}(t)/\Gamma(\alpha(t)) \qquad (2.4)$$

where $\bar{\beta}(t)$ is an entire function of t. This choice is by no means arbitrary and it is in fact necessitated if the correct Reggeization of amplitudes is to be accomplished without spurious singularities.[24] Notice that our parametrization carries an exponential dependence in t through the scale factor ν_1. Since the Γ function takes care of the appropriate kinematical factors, it will be

[24] We have explicitly checked our amplitudes following the method of G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) **46**, 239 (1968).

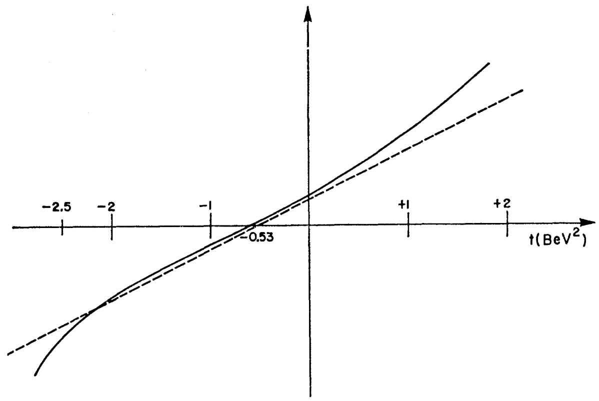


FIG. 2. Saturation of the $\pi\pi \to \pi\omega$ sum rules with the ρ resonance alone. Dashed line represents the resonance side and full line the Regge side. Ordinates in arbitrary units.

often assumed in the following that the function $\bar{\beta}(t)$ is a constant in the interval of t in which we are implementing the saturation. This is of course a dynamical assumption.

We use ν as the asymptotic variable, keeping in this way the s, u symmetry of the problem at every step. Regge behavior and analyticity requirements allow one to write the following family of sum rules:

$$\int_0^{\bar{\nu}} \nu^n \operatorname{Im}A(\nu,t)d\nu = \frac{\beta(t)}{\alpha(t)+n}\left(\frac{\bar{\nu}}{\nu_1}\right)^{\alpha(t)-1}\bar{\nu}^{n+1}. \quad (2.5)$$

First we concern ourselves with the lowest-moment nontrivial sum rule. Higher moments and Schwarz sum rules[22] are briefly discussed at the end of this section.

Hence we start studying Eq. (2.5) for $n=1$. This question was first studied in Ref. 14, but in this paper we go quite beyond that analysis. In particular we will study the t dependence of the equation as well as the question: to what an extent is the local average of the resonances given by the Regge term? We naturally start with the ρ meson alone in the resonances side. We must choose the cutoff between the values corresponding to the mass of the ρ meson and the mass of the first neglected particle along the trajectory ($J^P=3^-$). We assume, and this is supported by experiment, that if other trajectories exist they do not cross the $J=1$ line before this recurrence of the ρ. Extra contributions to the sum rules are discussed later. The arbitrariness on the choice of $\bar{\nu}$ is removed by the very stringent demand that the t dependence of both sides of the equation agrees over a larger region.

Using crossing, the ρ contribution can be expressed in terms of $\beta(m_\rho^2)$ as discussed in Appendix A and the sum rule reads

$$4\nu_\rho = \alpha(t)/\alpha'[(2\bar{\nu}\alpha')^2\Gamma^{-1}(\alpha+2)(k\bar{\nu}\alpha')^{\alpha-1}], \quad (2.6)$$

where

$$4\nu_\rho = 2m_\rho^2+t-\Sigma, \quad \Sigma = 3m_\pi^2+m_\omega^2, \quad \nu_1 = 1/k\alpha'. \quad (2.7)$$

Irrespective of the values of $\bar{\nu}$ and ν_1, Eq. (2.6) predicts a zero on the right-hand side at $t=-2m_\rho^2+\Sigma=-0.53$ BeV^2. The natural explanation of course is the zero of α that is responsible for the dip in πN charge exchange scattering. This was the main result of Ref. 14.[25] Since our equations are homogeneous, $\bar{\beta}$ drops out. Moreover we set

$$\bar{\nu} = \frac{1}{4\alpha'}(\alpha+\epsilon), \quad 0<\epsilon<4, \quad (2.8)$$

where ϵ expresses the range of possible values for $\bar{\nu}$. Equation (2.6) then reads

$$\nu_\rho = [\alpha(t)/4\alpha']\Phi_1(\alpha,\epsilon)(k/2)^{\alpha-1}, \quad (2.9)$$

where we have introduced

$$\Phi_1(\alpha,\epsilon) = (\tfrac{1}{2}\alpha+\tfrac{1}{2}\epsilon)^{\alpha+1}\Gamma^{-1}(\alpha+2). \quad (2.10)$$

We have assumed the trajectory to be linear at least in our region of interest and in accordance with the narrow resonance approximation.[13] The above condition $\alpha(-0.53)=0$ implies $\nu_\rho=\alpha/4\alpha'$, so that Eq. (2.9) becomes

$$1 = \Phi_1(\alpha,\epsilon)(k/2)^{\alpha-1}. \quad (2.11)$$

It is remarkable that the right-hand side of Eq. (2.11) is constant and equal to one to a high degree of accuracy in the region $-1<\alpha<1$, if we make the choice

$$\epsilon=2, \quad k=2. \quad (2.12)$$

The first condition selects the cutoff point to be exactly midway between the last resonance included and the first left out. This will turn out to be a general property of the equations. No other choice of the cutoff can fit the t dependence well. It is also a natural choice and we will use this criterion from now on. The choice $k=2$ determines our scale factor. With this determination of our two parameters, the two sides of Eq. (2.9) are plotted in Fig. 2. The agreement is unusually good in an extended region of t. We will discuss other properties of the parametrization in Sec. 3.[26]

Another important feature of the model is that, once the scale factor is chosen and the prescription for the displacement of the limit of integration is fixed, the inclusion of further resonances is possible without introducing further new parameters. This is clear since ν_1 cannot depend on the number of resonances included.

By means of Eq. (2.2) we can compute the contribution of the resonance of spin J lying on the Regge trajectory by going to $t=m_J^2$. Equation (2.2) actually gives only the leading contribution but one can easily compute the whole contribution that has to be propor-

[25] Notice, however, that we have changed the parametrization with respect to that paper since it was not suitable for introduction of more resonant states.

[26] To show the importance of the choice of parametrization we point out that, with our definitions, the value of k used by Mandelstam (Ref. 13) is $8/e$, i.e., our scale factor is $4/e$ times larger than his. Such a difference is enough to ruin the agreement presented in Fig. 2 quite considerably.

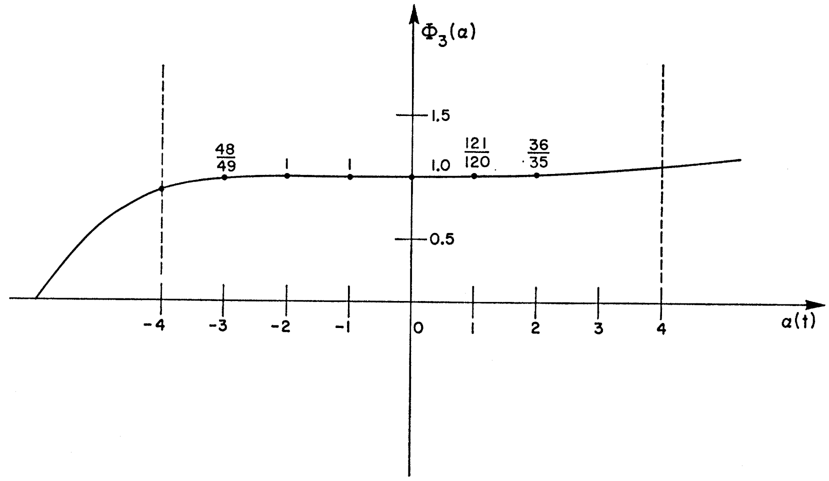


FIG. 3. Plot showing the typical behavior of a Φ function. The vertical dashed lines limit the region of constancy.

tional to $P_J'(\cos\theta)$. The proportionality constant can be determined by the leading term. Since the reaction is the same in all channels, it is then very easy to compute any resonance contribution to the sum rule. The general form of the sum rule, after inserting n resonances lying on the leading trajectory (supposed to be linear), is given by

$$\sum_{i=1}^{n} \frac{\bar{\beta}(m_i^2)}{(2i-2)!}(\alpha+\delta+4i-4)\tilde{P}_{2i-1}'(\alpha(t))=\bar{\beta}(t)\Phi_n(\alpha) \times\frac{\alpha(\alpha+2)(\alpha+3)\cdots(\alpha+2n-1)}{(2n-2)!}(k/2)^{\alpha-1}, \quad (2.13)$$

where we have defined

$$\Phi_n(\alpha)=\Phi_n(\alpha,4n-2)=(2n-2)!(\tfrac{1}{2}\alpha+2n-1)^{\alpha+1} \times\Gamma^{-1}(\alpha+2n) \quad (2.14)$$

and we have used, as previously explained,

$$4\bar{\nu}=m_n^2+m_{n+1}^2+t-\Sigma. \quad (2.15)$$

Moreover δ is a parameter defined by

$$\alpha(-2m_\rho^2+\Sigma)=\alpha(-0.53\ \mathrm{BeV}^2)=-\delta \quad (2.16)$$

and was found to be zero when the ρ alone was considered. Finally, $\tilde{P}_{2i+1}'(\alpha)$ is proportional to $P_{2i+1}'(\cos\theta_s)$ at $s=m_{i+1}^2$ but has the simpler asymptotic behavior

$$\tilde{P}_{21+1}'(\alpha) \underset{\alpha\to\infty}{\sim} \alpha^{2i}. \quad (2.17)$$

The function $\Phi_n(\alpha)$ turns out to be almost constant and equal to one in an interval increasing with n ($|\alpha|<2n$). For completeness we plot $\Phi_3(\alpha)$ in Fig. 3. The remaining terms on both sides of Eq. (2.13) are polynomials of the same degree. If we impose the condition that Eq. (2.13) be satisfied up to a slowly varying function of t, we must have $k=2$ independently of the number of resonances included, as expected, and also the midway prescription for the cutoff must hold at every step.

Next we assume $\bar{\beta}(t)=$ const. The first approximation ($n=1$) was discussed above and leasd to $\delta=0$, namely $\alpha(-0.53)=0$. The case $n=2$ demands the inclusion of the 3^- state. Equation (2.13) assumes the form

$$(\alpha+\delta)+\tfrac{1}{2}(\alpha+\delta+4)\tilde{P}_3'(\alpha) =\tfrac{1}{2}\Phi_2(\alpha)\times\alpha(\alpha+2)(\alpha+3). \quad (2.18)$$

Both sides of Eq. (2.18) are plotted in Fig. 4. The best value of δ is found to be -0.05, corresponding to $\alpha(-0.58)=0$. It is important to remember that in passing from Figs. 2 to 4 we had no free parameter available. The resonance side shows the three zeros near the desired points $\alpha=0$, -2, -3, and *none* around $\alpha=-1$. This is in fact the place where the zero of the Γ function of the Regge side has been erased by the integration. [See Eq. (2.5)].

The equality of the Regge and resonance sides holds now in a much larger region of t compared with the $n=1$ case. This is related of course to the remarkable property of the functions $\Phi_n(\alpha)$. It seems that the agreement is not accidental and that this step-by-step saturation provides for an iteration method to solve the sum rules. Notice also that the new resonances extend the region of validity of the equation, but do not alter significantly the results in the region where the first iteration was successful.

At this point one may hope that a single trajectory could bootstrap itself, as first conjectured by Mandelstam.[13] Moreover, since the function $\Phi_n(\alpha)$ has the property

$$\Phi_n(\alpha) \underset{\substack{n\to\infty \\ \alpha\ \text{fixed}}}{\longrightarrow} 1, \quad (2.19)$$

one could expect in that limit a mathematical solution valid for all t. This would accomplish the bootstrap of the linear trajectory. However, such a solution does not exist. One can see it by considering the cases $n=3$ and $n=4$. The resonance side of the sum rule cannot keep up with the Regge side. The resonances decrease too fast and the Regge amplitude can no longer be averaged by

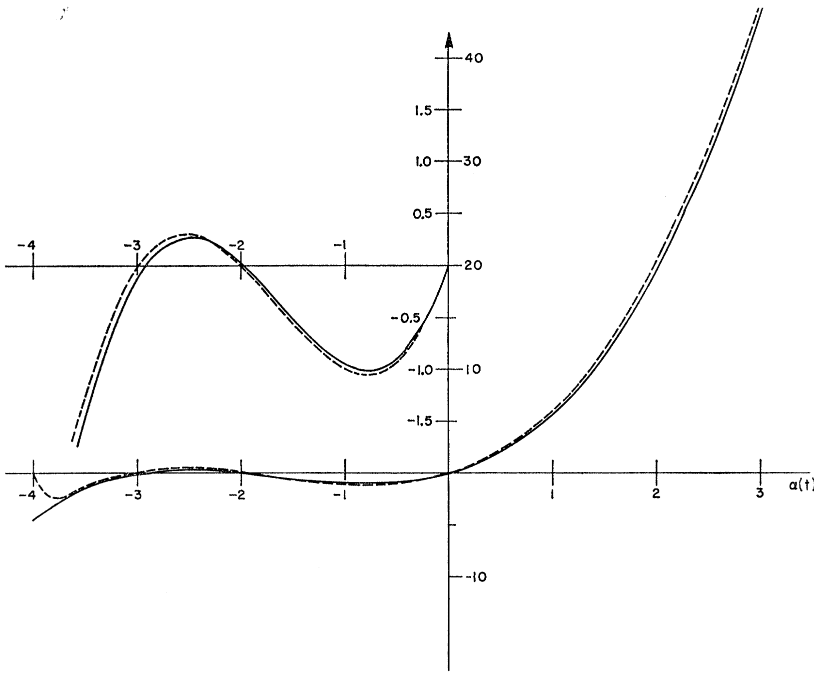


FIG. 4. Saturation of the same sum rule as in Fig. 2 with the ρ and the $\rho_3(3^-)$ in the resonance side. On the upper left side the most important region is shown on a larger scale. Here the Regge side is represented by the dashed line.

the resonances.[27] For $n=3$ this can be seen (Fig. 5) in the absence of the zeros at $\alpha=-4$ and -5 on the resonant side that should have been produced by the contribution of the 5^-. The Regge side, of course, possesses these zeros. The connection with the behavior of resonances of high spin, concerning their couplings to the external states, their dependence on t for the residue functions, and the experimental situation, is discussed in Sec. 3.

Several solutions may exist to the problems raised by the above result. In Ref. 15 we conjectured that the continuum may become important at higher energies and the resonance approximation may become inadequate. If this attitude is taken, one should choose a rather low value for the cutoff parameter and never include high-spin resonances on the right-hand side of Eq. (2.5). In order to get information on the pieces of the trajectory linked to high angular momenta, it seems that the spin of the external states has to be raised as well. We suggested[15] recently that a study of the family of reactions $\pi\pi\to\pi X_J$ (X_J being a normal-parity state of spin J) could provide the natural theoretical laboratory for studying the normal-parity trajectories. The first step of this program is the reaction $\pi\pi\to\pi\omega$ discussed in the present section. The next two steps are dealt with in Secs. 4 and 5.

In this section we want to attempt another solution, namely that resonances lying on nonleading trajectories are enough to provide for the contributions needed to reestablish the balance with the Regge part.

The existence of these trajectories is not an assumption; it is imposed by analyticity, since we are dealing with a scattering involving particles of unequal mass.

Analyticity by itself demands the existence of trajectories which are spaced by two units of angular momentum at $t=0$ and whose residues at this value of t are singular in such a fashion as to make the amplitude well-behaved. These singularities compensate the singularities appearing in nonleading terms of the parent-trajectory contribution.

Hence we need a dynamical model to make predictions about their behavior outside $t=0$. Some models, such as the Van Hove[4] or the Bethe-Salpeter[5] model, predict very different behavior for the parent and daughter trajectories. On the other hand, Toller *et al.*,[28] in recent studies based on $O(3,1)$, have found a solution demanding parallel trajectories away from $t=0$ as well. Clearly, this last possibility is very appealing for us since these daughters generate resonances at low energy that might help with the saturation. In order to study this possibility in the framework of our sum rules, we assume that the daughter trajectories are linear but we leave the slope α_D' and the residue function β_D as free parameters. Both are to be determined by requiring that, after extracting the $\Phi_n(\alpha)$ constant function, *the solution be mathematically exact.* We find a possible solution with parallel daughters. Though we cannot prove that the solution is unique, we consider it as an indication that the system can now possibly bootstrap itself. We parameterize our first daughter con-

[27] We are aware that there are more complicated possibilities such as having $\bar{\beta}(t)$ a polynomial. These results however do not depend on this choice.

[28] G. Cosenza, A. Sciarrino, and M. Toller, Trieste Report (unpublished) and Rome University Report, 1968 (unpublished).

tribution as follows:

$$A(\nu,t) \underset{\nu\to\infty}{\longrightarrow} \beta_D(t)\xi(\alpha_D)(\nu/\nu_1{}^D)^{\alpha_D-1}. \qquad (2.20)$$

By the same method we used before, we can compute the contributions from the resonances lying on the first daughter trajectory. The modified sum rule reads

$$\sum_{i=1}^{n}\frac{\bar\beta(m_i{}^2)}{(2i-2)!}(\alpha+\delta+4i-4)\tilde{P}_{2i-1}'(\alpha(t)) + \left(\frac{\alpha'}{\alpha_D'}\right)^2 \sum_{j=1}^{n'} \frac{(\alpha+\delta_D+4j-4)\beta_D(m_j{}^2)}{\sigma^{2j-2}} \times\tilde{P}_{2j-1}'(\alpha_D(t))+\cdots = \frac{\bar\beta(t)}{(2n-2)!}\alpha(\alpha+2)(\alpha+3)\cdots \times(\alpha+2n-1)\Phi_n(\alpha)+\cdots, \qquad (2.21)$$

where $\sigma=2\nu_1{}^D/\alpha_D'$ and δ_D is the analog of δ for the first daughter. We have also used $\nu_1=1/2\alpha'$. The dots on the left-hand side represent possible contributions arising from successive daughters, while those on the right-hand side are contributions of the nonleading regular terms of the Regge part obtained after cancellation of the unwanted $(t=0)$ singularities.

The cutoff $\bar\nu$ is still chosen according to the midway prescription. Correspondingly we have to include all the daughter contributions from resonances lying below $\bar\nu$. As we are not attempting here a systematic approach to the question of finding a general solution with a family of trajectories, we neglect for the moment the nonleading terms on the right-hand side. If we first saturate with the ρ, the sum rule gives ($\bar\beta=$const)

$$\alpha+\delta+\textstyle\sum_j=\alpha\Phi_1(\alpha). \qquad (2.22)$$

Hence, $\delta=0$ and the slope of the daughter must be such that no particle is produced before the cutoff. For $n=2$ we have

$$(\alpha+\delta)+\tfrac{1}{2}(\alpha+\delta+4)\{[\alpha+\tfrac{1}{2}(\delta+1)]^2-\tfrac{1}{5}\} +\textstyle\sum_j=\tfrac{1}{2}\alpha(\alpha+2)(\alpha+3)\Phi_2(\alpha). \qquad (2.23)$$

Since $\Phi_2\sim1$ and the α^3 coefficient is already the same on both sides, we need $\sum_j$ to make only lower-power contributions. Hence, the first 3^- particle on the daughter cannot appear before the cutoff. This is in fact a general feature of the equations. The leading terms are always the same when the leading trajectory alone is taken into account. Hence the daughter trajectories must be below the parent for $t\neq0$ as well.

The solution of Eq. (2.23) with $\Phi_2=1$ demands

$$\delta=0\,,\quad \textstyle\sum_j=-(\alpha+4)/40\,,\quad \delta_D=4\,, \qquad (2.24)$$

and this implies

$$\alpha_\rho(-2m_\rho{}^2+\Sigma)=0\,,\quad \alpha_D(t)=\alpha(t)-2\,,$$

and $\qquad (2.25)$

$$\beta_D(m_D{}^2)/\beta(m_\rho{}^2)=-1/40\,.$$

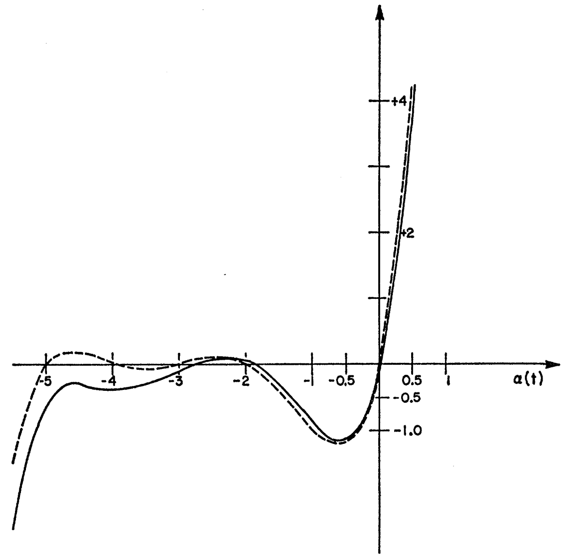


FIG. 5. Same as in Figs. 2 and 4 with the $\rho_5(5^-)$ included in the resonance side. Here the Regge side is represented by a dashed line.

So our equation *demands* that the first daughter be parallel to the leading trajectory. The small ratio of the residue function of the daughter to the leading one gives confidence in the previous saturation calculation, where the daughter was neglected.

Concerning the t dependence of $\beta_D(t)$ we cannot make too detailed predictions. We know that it must have a $1/t$ singularity at $t=0$ because of analyticity. If some assumption is made for the t dependence, we can determine the next to the leading term. Assume, e.g., that, after the singular term has been eliminated, the asymptotic expansion of $A(\nu,t)$ is of the form

$$\mathrm{Im}A(\nu,t) \underset{\nu\to\infty}{\longrightarrow} \bar\beta\Gamma^{-1}(\alpha)(\nu/\nu_1)^{\alpha-1} +\bar\beta_D\Gamma^{-1}(\alpha_D)(\nu/\nu_1)^{\alpha-3}. \qquad (2.26)$$

Then we predict $\bar\beta_D/\bar\beta=-\frac{1}{8}$.

As a consistency check, we can now feed back this term into the right-hand side of Eq. (2.23) and verify that the solution is essentially unchanged. This is indeed the case. Since the contribution of the daughter trajectory in Eq. (2.23) represents only a small correction, the determination of its slope is not, at this stage, very precise. However, the equality of the leading terms gives for α_D' an upper limit of about 1.5 BeV^{-2}.

As a next step we have included the s resonance lying on the parent (ρ) trajectory, consistently shifting the cutoff parameter. Once more we do not need any new 5^- state, but we need a 3^- state, provided by the first daughter, and a further 1^- particle demanding a second daughter. Notice that now the contribution of the daughters is crucial for the agreement, as one can see from Fig. 5. Once can easily see that the same procedure requires the second daughter to be parallel as

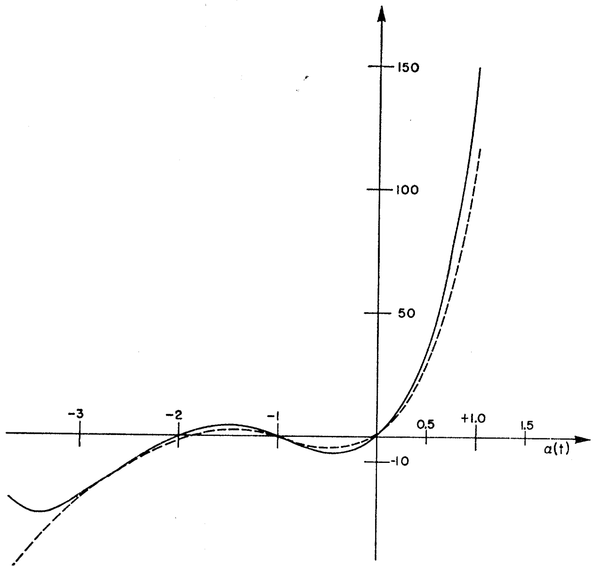


FIG. 6. Third moment sum rule for $\pi\pi \to \pi\omega$. Saturation with ρ and $\rho_3(3^-)$. Here the Regge side is represented by the full line.

well. It is important to remember that the solution is not trivial since the number of conditions is larger than the number of adjustable parameters. Further evidence for this family of parallel trajectories has been presented[18] by means of a Schmid analysis[19] of the Regge term, that confirms this picture in a rather impressive fashion. Extra restrictions coming from cancellation of singularities at half-integer points[29] for positive t make the saturation problem both interesting and complicated. It is, however, beyond the purpose of this paper, and we hope to come back to it in the future.

We look now into higher-moment and Schwarz sum rules. The next moment sum rule is

$$\int_0^{\bar\nu} \nu^3\, \mathrm{Im}A(\nu,t) = \frac{\beta(t)}{\alpha+3}(\bar\nu/\nu_1)^{\alpha(t)-1}\bar\nu^4. \qquad (2.27)$$

One can easily see that this sum rule cannot be saturated with the ρ alone, and be compatible with the previous sum rule. In fact the system of Eq. (2.5) with $n=1$ and 3 reads

$$\nu_\rho = \frac{\alpha(t)}{4\alpha'}\Phi(\alpha)\,, \qquad (2.28)$$

$$\nu_\rho{}^3 = \frac{\alpha(\alpha+1)}{4\alpha'(\alpha+3)}\Phi(\alpha)\bar\nu^2. \qquad (2.29)$$

These equations are inconsistent. Moreover, the second one has different t dependence on the two sides.

The features illustrated by this example are quite general: One needs always at least as many resonances as equations. This is most natural since high-moment sum rules emphasize high-mass resonances. Moreover, imposing more and more moment sum rules, corresponds to require that the average of the resonances made by the Regge form is more and more precise point by point. At the end, if sharp peaks exist at low energies, analyticity will require oscillations to exist up to the asymptotic region (eventually with smaller and smaller amplitude).

With this idea in mind, we try to satisfy the equations by including the contributions of the 3^-. Equation (2.28) transforms into Eq. (2.18) that holds very well. Equation (2.29) becomes

$$(\alpha+\delta)^3+\tfrac{1}{2}(\alpha+\delta+4)^3(\alpha+\tfrac{1}{2}\delta+\tfrac{1}{2}+\sqrt{\tfrac{1}{5}})(\alpha+\tfrac{1}{2}\delta+\tfrac{1}{2}-\sqrt{\tfrac{1}{5}}) = \tfrac{1}{2}\alpha(\alpha+1)(\alpha+2)(\alpha+6)^2\Phi_2(\alpha)\,. \qquad (2.30)$$

Equation (2.30) is plotted in Fig. 6 for $\delta=-0.1$ corresponding to $\alpha(-0.64)=0$. The agreement is quite satisfactory. Notice in particular that the resonance side shows, to a good approximation, the zeros at $\alpha=0$, -1, -2 and *not* at $\alpha=-3$, while the positions for the first-moment sum rule were 0, -2, and -3.

We look now into the Schwarz sum rules.[22] In our system they are supposed to hold for n even. As is well known, the presence of dips seems to be related to their validity since both depend on the smallness of the third double spectral function. In the case of πN charge-exchange scattering, there is good experimental evidence for the dip at the right position but the Schwarz sum rules are badly violated. Though there are possible accidental mechanisms to make dips present and Schwarz sum rules violated, this is an outstanding theoretical problem.[23] In our case the same result is obtained: The Schwarz sum rules are not satisfied. In fact, saturating the first Schwarz sum rule with the ρ, we obtain

$$1=\Gamma^{-1}(\alpha+1)\bar\nu\alpha'(\bar\nu/\nu_1)^{\alpha-1}=((\alpha+1)/4\bar\nu\alpha')\Phi_1(\alpha)\,, \quad \Phi_1(\alpha)\sim 1. \qquad (2.31)$$

Equation (2.31) demands $\bar\nu=(\alpha+1)/4\alpha'$ instead of our previous solution $\bar\nu=(\alpha+2)/4\alpha'$.

It is amusing to notice that the solution of sum rules demands such a cutoff as if a particle of spin 2 could couple to the reaction and would lie on a degenerate trajectory. Such a trajectory cannot exist because of crossing symmetry and conversely crossing symmetry makes the sum rule trivial when $\rho_{su}\neq 0$.

Consider finally the possibility of saturating the system of equations for the moments $n=1$, 2, and 3, saturating with ρ and 3^-. One finds the necessary conditions

$$\nu_\rho\sim\alpha_\rho\,, \quad \nu_R\sim\alpha+2\,, \quad \bar\nu\sim\alpha+2. \qquad (2.32)$$

Equation (2.32) implies that the mass of the 3^- is where the trajectory crosses $J=2$ and the cutoff is the one corresponding to such a mass.

[29] The possibility that these same daughters cancel these singularities is very appealing. However the problem requires further study. See, for example, K. Dietz, J. Honerkamp, and J. Kupsh, University of Bonn Report, 1968 (unpublished).

The failure of the Schwarz sum rules poses a delicate problem. However, it seems clear that the correspondence between the average amplitude and the Regge amplitude is quasilocal and that no compensation occurs from points that are very distant in ν. It seems possible that the local average holds in the ν variable even when some contributions are coming from the other channel, a most intriguing and interesting possibility. This seems to be the case in $\pi\rho$ scattering and in other processes as well.

3. EXPERIMENTAL AND THEORETICAL CONSIDERATIONS ON $\beta(t)$

A. Theoretical

In all the reactions discussed in the paper (see also Secs. 4 and 5) we have assumed linear trajectories only in a region around $t=0$. Our results do not imply, neither do they necessitate, the linearity of the trajectories when the energy goes to infinity. Nevertheless, it is interesting to make the assumption of a kind of maximal simplicity and to suppose both linearity of the trajectories and validity of our parametrization of the β residue function at all energies.

Khuri[30] has shown that rising trajectories imply very restrictive conditions on the Regge residue functions $\beta(t)$ such as to force them not to obey dispersion relations. Jones and Teplitz[31] have subsequently argued that it is plausible for $\beta(t)$ to have an essential singularity at infinity for rising trajectories, thus restoring consistency with Khuri's theorem. Here we want to see how Khuri's paradox is solved in the context of this work.

The main restriction imposed by Khuri can be stated as follows: The Regge term in the s channel is bounded by a polynomial in s for $s\rightarrow+\infty$ both at fixed t and fixed $z=\cos\theta_s$.

Consider the reaction $\pi\pi\rightarrow\pi\omega$. We neglect for simplicity the pion mass. By means of a partial-wave expansion and a Sommerfeld-Watson transformation we can write the s-channel Regge-pole contribution as

$$A(\nu,t)=-\left(\frac{2}{s\pi}\right)^{1/2}\frac{(2\alpha+1)}{q_sp_s}\frac{\hat{\beta}(s)}{[\alpha(\alpha+1)]^{1/2}}\frac{(e^{-i\pi\alpha}-1)}{\sin\pi\alpha}\times\frac{\alpha\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}2^{\alpha-1}z^{\alpha-1},\quad(3.1)$$

where $\hat{\beta}(s)$ is the residue of the Regge pole of the analytically continued partial-wave amplitude with definite helicity $f_+(J,s)$:

$$f_+(J,s)\simeq\hat{\beta}(s)/[J-\alpha(s)]\quad(3.2)$$

and

$$p_s=\tfrac{1}{2}\sqrt{s},\quad q_s=(s-m^2)/2\sqrt{s}.\quad(3.3)$$

[30] N. N. Khuri, Phys. Rev. Letters 18, 1094 (1967).
[31] C. E. Jones and V. Teplitz, Phys. Rev. Letters 135, 29 (1967).

Our parametrization of $A(\nu,t)$, Eq. (2.2), implies

$$\frac{\hat{\beta}(s)}{s^{1/2}}\frac{(2\alpha+1)}{[\alpha(\alpha+1)]^{1/2}}\left(\frac{2}{\pi}\right)^{1/2}\frac{\Gamma(\alpha+\frac{1}{2})}{2(q_sp_s)^\alpha}2^\alpha\nu_1{}^{\alpha-1}=\bar{\beta}(s),\quad(3.4)$$

where $\bar{\beta}(s)$ is known to be smooth function at low energy and is bounded by a polynomial at all energies. Then

$$\hat{\beta}(s)=\bar{\beta}(s)\left(\frac{\pi}{2}\right)^{1/2}\tfrac{1}{2}\nu_1s^{1/2}\frac{[\alpha(\alpha+1)]^{1/2}}{\Gamma(\alpha+\frac{3}{2})}\left(\frac{q_sp_s}{2\nu_1}\right)^\alpha.\quad(3.5)$$

Mandelstam's parametrization is different than ours. A similar analysis yields

$$\hat{\beta}(s)_{\text{Mandelstam}}\underset{s\to\infty}{\longrightarrow}\text{polynomial in } s.$$

His scale factor is $\nu_1=e/8\alpha'$, while we have found $\nu_1=1/2\alpha'$, which is a little bit less than twice Mandelstam's one. From Eq. (3.5) we see that our residue satisfies

$$\lim_{s\to+\infty}\hat{\beta}(s)=(\tfrac{1}{4}e)^{\alpha(s)}P(s)=e^{-0.4s[\text{GeV}^2]}P(s),\quad(3.6)$$

$$\lim_{s\to-\infty}\hat{\beta}(s)=(\tfrac{1}{4}e)^{\alpha(s)}P(s)=e^{0.4|s|[\text{GeV}^2]}P(s),\quad(3.7)$$

where $P(s)$ is bounded by a polynomial.

We next prove that Eq. (3.5) implies the Khuri bounds to be satisfied. These read

$$|(2\alpha+1)\hat{\beta}(s)/[\sin\pi\alpha(s)]d_{01}{}^\alpha(z)|<s^N$$
$$\text{when } s\rightarrow+\infty \text{ and } z \text{ or } t \text{ is fixed.}\quad(3.8)$$

Let us first consider z fixed. We use the relation

$$d_{01}{}^\alpha(z)=\frac{(1-z^2)^{1/2}}{[\alpha(\alpha+1)]^{1/2}}\frac{d}{dz}P_\alpha(z)\quad(3.9)$$

and, for fixed physical θ_s

$$\lim_{J\to\infty}P_J(z)\simeq\left(\frac{2}{\pi J}\right)^{1/2}\frac{\sin[(J+\frac{1}{2})\theta_s+\frac{1}{4}\pi]}{(\sin\theta_s)^{1/2}},\quad(3.10)$$

$$\text{while for } \theta_s=0 \text{ we use } P_J(1)=1.\quad(3.11)$$

Then

$$|P_{\alpha(s)}(z)|<\text{const},\quad(3.12)$$

where $\alpha\rightarrow\infty$ and z is fixed. A similar argument can be used for $d_{01}{}^\alpha$ (z). Finally,

$$\left|\frac{(2\alpha+1)\hat{\beta}(s)d_{01}{}^\alpha(z)}{\sin\pi\alpha(s)}\right|\leq\left|\frac{\hat{\beta}(s)}{\sin\pi\alpha}\right|Cs=C\frac{e^{-0.4s[\text{GeV}^2]}P(s)}{|\sin\pi\alpha|}.\quad(3.13)$$

The inequality (3.8) cannot hold at the particle pole. This difficulty can be avoided by relaxing the narrow-resonance approximation and allowing for a small nonzero imaginary part at the pole. Alternatively, the inequality can be interpreted as referring to local averages in some interval.

Clearly our result (3.13) shows that we obey the inequality.

Consider now the fixed-t limit. Then,

$$P_{\alpha(s)}(1+2t/s-m^2)\sim I_0(2\alpha\sqrt{t}/(s-m^2)^{1/2}). \quad (3.14)$$

Since $\alpha(s)=\alpha' s+b$, we have, as $s\to\infty$, (3.15)

$$P_{\alpha(s)}\left(1+\frac{2t}{s-m^2}\right)\sim[4\pi\alpha'(st)^{1/2}]^{-1/2}\exp[2\alpha'(st)^{1/2}]. \quad (3.16)$$

Then

$$d^{\alpha}{}_{01}(z)\sim\left[1-\left(1+\frac{2t}{s-m^2}\right)^2\right]^{1/2}\times\frac{1}{[\alpha(\alpha+1)]^{1/2}}\frac{dt}{dz}\frac{d}{dt}P_{\alpha(s)} \quad (3.17)$$

$$=[4\pi\alpha'(st)^{1/2}]^{-1/2}\exp[2\alpha'(st)^{1/2}] \quad (3.18)$$

as $s\to\infty$ with t fixed. Finally

$$|(2\alpha+1)\hat{\beta}(s)d_{01}{}^{\alpha}(z)|\leq\frac{1}{(4\pi\alpha' st)^{1/2}}\exp[-0.4s+2\alpha'(st)^{1/2}]\underset{s\to+\infty}{\to}0. \quad (3.19)$$

The reason why we are able to satisfy both bounds is that our parametrization of the β forces the latter to have an essential singularity at infinity. It is also important[30] that the trajectories grow faster than $s^{1/2}$ and in particular linearly for the fixed-angle bound. Hence our solution of the Khuri paradox is identical to the one proposed by Jones and Teplitz.[31] Our β, however, decays exponentially and does not go like the inverse of a Γ function as in their case.

B. Experimental

We discuss here the experimental consequences of our parametrization. We start by recalling that trajectory functions are in good agreement with the assumption of linearity at low energies in the particle side and in the high-energy low-t scattering region. Trajectory degeneracy seems well established to some 20% or better.

Our main task here, however, is to discuss the $\beta(t)$ residue function. Our definition of the scale parameter turns out to coincide with that of Kramer and Maor[32] for high-energy fits. We then have

$$\nu_1=s_0=(1/2\alpha')\simeq 0.6\ \text{GeV}^2. \quad (3.20)$$

Assuming constancy of the residue function, apart from the Γ function, we have a definite exponential t dependence induced by the scale factor.

Let us first discuss the negative region. The difference between a ν^{α} and an s^{α} asymptotic behavior can be tested by performing large-momentum-transfer (fixed-angle) scattering. We predict the existence of secondary dips. Kramer and Maor[32] have obtained good fits for inelastic high-energy scattering involving the vector and tensor trajectories with a parametrization very similar to ours. In particular, their scale parameter is very close to that of Eq. (3.20). These authors have also found necessary, in order to fit the data, direct Reggeization of invariant amplitudes rather than helicity amplitudes. From the point of view of our sum rules, it looks quite difficult to match the extra t dependence introduced by the t-dependent crossing matrix of the helicity amplitudes.

Also we predict no dip for the A_2 trajectory (Gell-Mann mechanism) in agreement with other calculations.[33]

The values of the constants $\bar{\beta}$, required by our solution, demand relations between the different trajectories and in particular, as explained in Sec. 4, we have degeneracy of the residue functions in agreement with the analysis of Ref. 32. A more direct test of our $\beta(t)$ could be achieved by using factorization in the context of the reactions. πN, NN scattering, and $\pi N\to\omega N$. However, the experiments are still too imprecise to let us draw any conclusion.

Concerning the positive-t region, we already stated our prediction of an exponential dependence of the couplings as a function of J. Such a law seems verified[34] in the experimentally accessible case of baryonic Regge recurrences. For mesons we can predict, from the t dependence of $\beta(t)$, the ratio

$$\frac{g_{R\pi\pi}g_{R\pi\omega}}{g_{\rho\pi\pi}g_{\rho\pi\omega}}=\frac{1}{16}\frac{\beta(m_3{}^2)}{\beta(m_1{}^2)}\times\frac{1}{\nu_1{}^2}=0.87\times10^{-1}\ \text{BeV}^{-4}, \quad (3.21)$$

where $g_{\rho\pi\pi}$ and $g_{\rho\pi\omega}$ are the conventionally defined ρ couplings and we used

$$\mathcal{L}(R\pi\pi)=ig_{R\pi\pi}e_{\mu\nu\rho}{}^{(R)}(p_1-p_2)_{\mu}(p_1-p_2)_{\nu}(p_1-p_2)_{\rho}, \quad (3.22)$$

$$\mathcal{L}(R\omega\pi)=ig_{R\pi\omega}\epsilon_{\mu\nu\rho\sigma}e_{\mu\lambda\eta}{}^{(R)}e_{\nu}{}^{(\omega)}q_{\rho}p_{3\sigma}\times(q-p_3)_{\lambda}(q-p_3)_{\eta}. \quad (3.23)$$

The kinematics are defined as for an s-channel R exchange in Fig. 1. Equation (3.21) puts only a lower limit on the R width, and is completely consistent with present data.

4. STUDY OF THE SUM RULES FOR $\pi\pi\to\pi A_2$

There are several interesting new features in this reaction, which can be thought of as a second step in the study of the general $PP\to PJ$ scattering (see Sec. 1). First, there are now two independent helicity ampli-

[32] M. Kramer and U. Maor (to be published). We thank Uri Maor for giving us his results prior to publication and for interesting discussions on the subject.

[33] F. Gilman, H. Harari, and Y. Zarmi, Phys. Rev. Letters 21, 323 (1968).

[34] H. Goldberg, Phys. Rev. Letters 19, 1391 (1967).

tudes, since, though the helicity zero of any normal-parity state will remain uncoupled, the spin 2 of the external state allows for a new possibility. Second, the isospin structure of the system is now more complicated and allows more states to contribute in all channels.

As shown in Appendix A (see also Ref. 15), the $I=0$ Pomeranchuk trajectory that controls high-energy scattering does not couple to our reaction. This was shown to be the case in the $SU(3)$ limit. Since experimentally diffractive production of the A_2 is not observed, this is a good indication that the πA_2 Pomeranchuk vertex is indeed very small.[35]

Even if it is not zero, Harari[21] has presented arguments suggesting that the Pomeranchuk trajectory is generated by the continuum and not by the resonances; hence we shall ignore the Pomeranchuk trajectory in any case.

Because of the presence of two helicity states, the choice of the amplitudes becomes a nontrivial problem. The point is that helicity and invariant amplitudes have extra t-dependent factors with respect to each other. The former must in fact obey conspiracy conditions at the thresholds and pseudothresholds of the t channel in order to insure conservation of angular momentum.

This problem did not appear in the previous example ($\pi\pi \longrightarrow \pi\omega$) because only one independent amplitude is present there.

In the case under study in this section, the problem can be stated as knowing which amplitudes show a simple asymptotic behavior like that of Eq. (2.2) with a smooth $\bar{\beta}(t)$.

Notice that, as we need only the leading term in ν, we can think equivalently in terms of Regge or Khuri poles. In the first case use of regularized helicity amplitudes would seem preferable, while the invariant amplitudes seem the natural choice for Khuri poles.

We notice, however, that the conspiracy relations forbid a very simple dependence like that of Eq. (2.2) for the helicity amplitudes. We then choose to work with the invariant amplitudes. In this way conspiracy relations are automatically satisfied.[36]

Still we shall use the helicity amplitudes as an intermediate tool, in order to calculate the contributions of each resonance to the sum rule while avoiding the cumbersome propagator techniques for high-spin particles.

We shall consider saturation of the sum rules with resonating states up to $J=4$ and lying on the leading trajectories.

Hence our task is to compute the contribution to the two invariant amplitudes of the following particles: $\rho(1^-)$, $f(2^+)$, $\rho_3(3^-)$, and $f_4(4^+)$.

[35] D. Morrison, Phys. Rev. **165**, 1699 (1968).

[36] It is interesting to point out that our amplitudes obey a conspiracy condition at both threshold and pseudothreshold, and not evasion. We have no kinematical constraint at $t=0$ (in our case an unphysical point). These constraints, however, cannot be tested experimentally.

Our scattering amplitude is decomposed as (see Fig. 1 for the kinematics)

$$T^{\alpha\beta\gamma\delta}=i\epsilon_{\mu\phi\lambda\sigma}e_{\mu\nu}{}^{(A_2)}p_{1\phi}p_{2\lambda}p_{3\sigma} \times[(p_2+p_3)_\nu A^{\alpha\beta\gamma\delta}+(p_2-p_3)_\nu B^{\alpha\beta\gamma\delta}]. \quad (4.1)$$

The crossing properties

$$A^{(I)}(-\nu)=(-)^I A^{(I)}(\nu),$$
$$B^{(I)}(-\nu)=(-)^{I+1}B^{(I)}(\nu) \quad (4.2)$$

can be easily verified. In (4.2), I is the isospin in the t channel. To compute the contributions of the resonances to our equations, we introduce helicity amplitudes.

The t-channel helicity amplitudes are seen to be related to the invariant ones by

$$T_1{}^t=A[-q_0(q\sqrt{t}/tm)ip_t{}^2\sin\theta_t\cos\theta_t] +B[q_t{}^2[(-i)/m]p_t\sin\theta_t\tfrac{1}{2}t] \quad (4.3)$$

and

$$T_2{}^t=-[ip_t{}^2q_t(\sqrt{t})\sin^2\theta_t]A\,, \quad (4.4)$$

where (m is the A_2 mass)

$$t=4p_t{}^2+4m_\pi{}^2\,,\quad q_t{}^2=(1/4t)[(t-m^2-m_\pi{}^2)^2-4m^2m_\pi{}^2]\,,$$
$$q_{0t}=(1/2\sqrt{t})(t+m^2-m_\pi{}^2).$$

As usual, one defines the singularity-free helicity amplitudes by

$$\tilde{T}_1{}^t=2(\sin\theta_t)^{-1}T_1{}^t\,, \quad (4.5)$$

$$\tilde{T}_2{}^t=4(\sin^2\theta_t)^{-1}T_2{}^t. \quad (4.6)$$

A similar procedure can be followed in the s channel and, in this way, amplitudes that are free of kinematical singularities and zeros in both the t and s channel can be constructed. Because of the crossing properties (4.2), the lowest-moment nontrivial sum rules turn out to be seven, namely,

$$\int \mathrm{Im}A^{(1)}d\nu=0\,, \quad (4.7)$$

$$\int \nu\,\mathrm{Im}B^{(1)}=0\,, \quad (4.8)$$

$$\int \nu\,\mathrm{Im}A^{(0,2)}d\nu=0\,, \quad (4.9,4.10)$$

$$\int \mathrm{Im}B^{(0,2)}d\nu=0\,, \quad (4.11,4.12)$$

$$\int \nu^2\,\mathrm{Im}A^{(1)}=0. \quad (4.13)$$

In order to evaluate the resonance contribution, we have to express A and B in terms of the s-channel helicity amplitudes. This could be achieved by using Eqs. (4.3)–(4.6) and then the crossing matrix between s- and t-channel helicity amplitudes.

TABLE I. Values of c_i in Eq. (4.30) and form of the right-hand side.

Sum rule	ρ_3 c_1	f c_2	c_3	$\rho_3(3^-)$ c_4	c_5	$f(4^+)$ c_6	c_7	Right-hand side
1	1	1	1	1	1	1	1	$\frac{\bar{\beta}_\rho{}^A}{\Gamma(\alpha_\rho-1)}\frac{\bar{\nu}}{\alpha_\rho-1}\left(\frac{\bar{\nu}}{\nu_1}\right)^{\alpha\rho-2}$
2	$-\nu_\rho$	$-\nu_f$	$3\nu_f$	$-\nu_{3^-}$	$3\nu_{3^-}$	$-\nu_{4^+}$	$3\nu_{4^+}$	$\frac{\bar{\beta}_\rho{}^B}{\Gamma(\alpha_\rho)}\frac{\bar{\nu}^2}{\alpha_\rho+1}\left(\frac{\bar{\nu}}{\nu_1}\right)^{\alpha\rho-1}$
3	$2\nu_\rho$	0	0	$2\nu_{3^-}$	$2\nu_{3^-}$	0	0	$\frac{\bar{\beta}_f{}^A}{\Gamma(\alpha_f-1)}\frac{\bar{\nu}^2}{\alpha_f}\left(\frac{\bar{\nu}}{\nu_2}\right)^{\alpha_f-2}$
4	0	$2\nu_f$	$2\nu_f$	0	0	$2\nu_{4^+}$	$2\nu_{4^+}$	$\frac{\bar{\beta}_f{}^A}{\Gamma(\alpha_f-1)}\frac{\bar{\nu}^2}{\alpha_f}\left(\frac{\bar{\nu}}{\nu_2}\right)^{\alpha_f-2}$
5	-2	0	0	-2	6	0	0	$\frac{\bar{\beta}_f{}^B}{\Gamma(\alpha_f)}\frac{\bar{\nu}}{\alpha_f}\left(\frac{\bar{\nu}}{\nu_2}\right)^{\alpha_f-1}$
6	0	-2	6	0	0	-2	6	$\frac{\bar{\beta}_f{}^B}{\Gamma(\alpha_f)}\frac{\bar{\nu}}{\alpha_f}\left(\frac{\bar{\nu}}{\nu_2}\right)^{\alpha_f-1}$
7	$\nu_\rho{}^2$	$\nu_f{}^2$	$\nu_f{}^2$	$\nu_{3^-}{}^2$	$\nu_{3^-}{}^2$	$\nu_{4^+}{}^2$	$\nu_{4^+}{}^2$	$\frac{\bar{\beta}_\rho{}^A}{\Gamma(\alpha_\rho-1)}\frac{\bar{\nu}^3}{\alpha_\rho+1}\left(\frac{\bar{\nu}}{\nu_1}\right)^{\alpha\rho-2}$

However, it is simpler to introduce s-channel invariant amplitudes, obtained by Eq. (4.1) after the substitution $p_1 \leftrightarrow -p_3$, and use crossing symmetry at the invariant-amplitudes level, where the crossing matrix is numerical and immediately computed to be

$$C_{\text{inv}\cdot\text{ampl}.}{}^{s,t}=\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & +\frac{1}{2} \end{pmatrix}. \tag{4.14}$$

TABLE II. Values of K_i in Eq. (4.30).

i	K_i
1	$-\frac{1}{4}\frac{\bar{\beta}_\rho{}^B}{\alpha_\rho'}$
2	$-\frac{1}{4}\frac{\bar{\beta}_f{}^B}{\alpha_f'}\left(\frac{p_fq_f}{\nu_2}\right)z_f$
3	$-\frac{1}{4}\frac{\bar{\beta}_f{}^A}{\alpha_f'}$
4	$-\frac{1}{8}\frac{\bar{\beta}_\rho{}^B}{\alpha_\rho'}\left(\frac{p_3{}^-q_3{}^-}{\nu_1}\right)^2 z_3{}^{-2}+\frac{1}{40}\frac{1}{\alpha_\rho'}\times\left[\bar{\beta}_\rho{}^B\left(\frac{p_3{}^-q_3{}^-}{\nu_1}\right)^2+2\bar{\beta}_\rho{}^A\frac{2q_{03}{}^-p_3{}^{-2}}{m_R\nu_1}\right]$
5	$-\frac{1}{4}\frac{\bar{\beta}_\rho{}^A}{\alpha_\rho'}\frac{p_3{}^-q_3{}^-}{\nu_1}z_3{}^-$
6	$-\frac{1}{24}\frac{\bar{\beta}_f{}^B}{\alpha_f'}\left(\frac{p_4{}^+q_4{}^+}{\nu_2}\right)^3 z_f{}^{*3}+\frac{1}{7\alpha_f'}\times\left[\frac{1}{8}\bar{\beta}_f{}^B\frac{p_4{}^+q_4{}^+}{\nu_2}+\frac{1}{2}\bar{\beta}_f{}^A\frac{p_4{}^+q_4{}^+}{m_4{}^+q_4{}^+}\right]\left(\frac{p_4{}^+q_4{}^+}{\nu_2}\right)^2 z_4{}^+$
7	$-\frac{3}{24}\frac{\bar{\beta}_f{}^A}{7\alpha_f'}\left(\frac{p_4{}^+q_4{}^+}{\nu_2}\right)^2(7z_4{}^{+2}-1)$

As a final result we get

$$A=-\tfrac{1}{2}im(1/p_sq_s{}^2s)\tilde{T}_1{}^s+(i/4p_s{}^2q_s{}^2s)(p_sq_{0s}\cos\theta_s-\tfrac{1}{2}q_s\sqrt{s})\tilde{T}_2{}^s=\sum_\lambda a_\lambda(s,t)\tilde{T}_\lambda{}^s, \tag{4.15}$$

$$B=\tfrac{1}{2}im(1/p_sq_s{}^2s)\tilde{T}_1{}^s-(i/4p_s{}^2q_s{}^2s)(p_sq_{0s}\cos\theta_s+\tfrac{3}{2}q_s\sqrt{s})\tilde{T}_2{}^s=\sum_\lambda b_\lambda(s,t)\tilde{T}_\lambda{}^s. \tag{4.16}$$

These formulas are enough to enable us to write

$$T_i{}^t=c_{ij}T_j{}^s. \tag{4.17}$$

The sum rules can now be written down also for the regularized helicity amplitudes.

The integral over the resonances can now be calculated. We have

$$A^{(I)}=\sum_{I'}C_{II'}\sum_\lambda a_\lambda(s,t)\tilde{T}_\lambda{}^{(s,I')}, \tag{4.18}$$

$$B^{(I)}=\sum_{I}C_{II'}\sum_\lambda b_\lambda(s,t)\tilde{T}_\lambda{}^{(s,I')}, \tag{4.19}$$

where $C_{II'}$ is the $SU(2)$ crossing matrix. If we define

$$T_{\alpha\beta\gamma\delta}=\delta_{\beta\gamma}\delta_{\alpha\delta}T_t{}^{(0)}+\epsilon_{\beta\gamma\xi}\epsilon_{\alpha\delta\xi}T_t{}^{(1)}+(\delta_{\beta\alpha}\delta_{\gamma\delta}+\delta_{\gamma\alpha}\delta_{\beta\delta}-\tfrac{2}{3}\delta_{\beta\gamma}\delta_{\alpha\delta})T_t{}^{(2)}, \tag{4.20}$$

TABLE III. Functions Φ and forms of the right-hand side of Eq. (4.30) after saturation with ρ and f (first two columns) and ρ, f, $\rho_3(3^-)$, and $f_4(4^+)$ (last two columns).

Sum rule No.	Saturation with ρ and f: Smooth function	Saturation with ρ and f: Right-hand side with $\Phi=1$	Saturation with ρ, f, $\rho_3(3^-)$, and $f_4(4^+)$: Smooth function	Saturation with ρ, f, $\rho_3(3^-)$, and $f_4(4^+)$: Right-hand side with $\Phi=1$
1	$\Phi(\alpha)=\Gamma^{-1}(\alpha+1)[(\alpha+2)/2]^{\alpha-1}$	$\nu_1\bar{\beta}_\rho{}^A\cdot\alpha_\rho(t)$	$\Phi(\alpha)=3!\Gamma^{-1}(\alpha+3)[(\alpha+6)/2]^{\alpha-1}$	$\frac{1}{6}\bar{\beta}_\rho{}^A\nu_1\alpha_\rho(\alpha_\rho+1)(\alpha_\rho+2)$
2	$\Phi(\alpha)=\Gamma^{-1}(\alpha+2)[(\alpha+2)/2]^{\alpha+1}$	$\nu_1{}^2\bar{\beta}_\rho{}^B\cdot\alpha_\rho(t)$	$\Phi(\alpha)=2\Gamma^{-1}(\alpha+4)[(\alpha+6)/2]^{\alpha+1}$	$\frac{1}{2}\bar{\beta}_\rho{}^B\nu_1{}^2\alpha_\rho(\alpha_\rho+2)(\alpha_\rho+3)$
3	$\Phi(\alpha)=\Gamma^{-1}(\alpha+1)[(\alpha+1)/2]^{\alpha}$	$\nu_2{}^2\bar{\beta}_f{}^A[\alpha_f(t)-1]$	$\Phi(\alpha)=2\Gamma^{-1}(\alpha+3)[(\alpha+5)/2]^{\alpha}$	$\frac{1}{2}\bar{\beta}_f{}^A\nu_2{}^2(\alpha_f-1)(\alpha_f+1)(\alpha_f+2)$
4	$\Phi(\alpha)=\Gamma^{-1}(\alpha+2)[(\alpha+3)/2]^{\alpha}$	$\nu_2{}^2\bar{\beta}_f{}^A[\alpha_f(t)-1][\alpha_f(t)+1]$	$\Phi(\alpha)=3!\Gamma^{-1}(\alpha+4)[(\alpha+7)/2]^{\alpha}$	$\frac{1}{6}\bar{\beta}_f{}^A\nu_2{}^2(\alpha_f-1)(\alpha_f+1)(\alpha_f+2)(\alpha_f+3)$
5	$\Phi(\alpha)=\Gamma^{-1}(\alpha+1)[(\alpha+1)/2]^{\alpha}$	$\nu_2\bar{\beta}_f{}^B$	$\Phi(\alpha)=2\Gamma^{-1}(\alpha+3)[(\alpha+5)/2]^{\alpha}$	$\frac{1}{2}\bar{\beta}_f{}^B\nu_2(\alpha_f+1)(\alpha_f+2)$
6	$\Phi(\alpha)=\Gamma^{-1}(\alpha+2)[(\alpha+3)/2]^{\alpha+1}$	$\nu_2\bar{\beta}_f{}^B[\alpha_f(t)+1]$	$\Phi(\alpha)=3!\Gamma^{-1}(\alpha+4)[(\alpha+7)/2]^{\alpha+1}$	$\frac{1}{6}\bar{\beta}_f{}^B\nu_2(\alpha_f+1)(\alpha_f+2)(\alpha_f+3)$
7	$\Phi(\alpha)=\Gamma^{-1}(\alpha+2)[(\alpha+2)/2]^{\alpha+1}$	$\nu_1{}^3\bar{\beta}_\rho{}^A[\alpha_\rho(t)-1]\alpha_\rho(t)$	$\Phi(\alpha)=2\Gamma^{-1}(\alpha+4)[(\alpha+6)/2]^{\alpha+1}$	$\frac{1}{2}\bar{\beta}_\rho{}^A\nu_1{}^3\alpha_\rho(\alpha_\rho-1)(\alpha_\rho+2)(\alpha_\rho+3)$

we find

$$C_{II'}=\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & 10/9\\ \frac{1}{2} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{6}\end{bmatrix}. \qquad (4.21)$$

The Regge residue functions are defined by the equations (valid for large ν and fixed t)

$$A^{(1)}=\xi^-(\alpha_\rho)\beta_\rho{}^A(t)(\nu/\nu_1)^{\alpha_\rho-2}, \qquad (4.22)$$

$$A^{(0)}=-\xi^+(\alpha_f)\beta_f{}^A(t)(\nu/\nu_2)^{\alpha_f-2}, \qquad (4.23)$$

$$B^{(1)}=\xi^-(\alpha_\rho)\beta_\rho{}^B(t)(\nu/\nu_1)^{\alpha_\rho-1}, \qquad (4.24)$$

$$B^{(0)}=-\xi^+(\alpha_f)\beta_f{}^B(t)(\nu/\nu_2)^{\alpha_f-1}, \qquad (4.25)$$

where

$$\xi^\pm(\alpha_i)=(1\pm e^{-i\pi\alpha_i})/\sin\pi\alpha_i, \qquad (4.26)$$

$$\beta_{\rho,f}{}^A(t)=\Gamma^{-1}(\alpha-1)\bar{\beta}_{\rho,f}{}^A, \qquad (4.27)$$

$$\beta_{\rho,f}{}^B(t)=\Gamma^{-1}(\alpha)\bar{\beta}_{\rho,f}{}^B, \qquad (4.28)$$

and

$$\bar{\beta}^A,\quad \bar{\beta}^B\simeq\text{const.} \qquad (4.29)$$

The reader is in possession of all means needed to write the sum rules. The road is straightforward, but the algebraic calculations are lengthy. One of the seven equations is derived in Appendix B to show the method in detail. We thus obtain the system of sum rules:

$$\sum_i c_iK_i=\text{ right-hand side}. \qquad (4.30)$$

The coefficients c_i and the right-hand sides (Regge contribution) are given in Table I for the various sum rules, while the expressions for the K_i's are listed in Table II.

We start by saturating the resonance side with ρ and f only. The sum over i in Eq. (4.30) extends then up to 3. Furthermore, by following the procedure of Sec. 2, it is possible to extract from the right-hand side of each of the seven sum rules a smooth and practically constant function $\Phi_n(\alpha)$. The Φ functions as well as the final form of the right-hand side are given in Table III. Letting now $\Phi=1$, we get an algebraic system of seven equations that are immediately read from Tables I and III.

From sum rule 3 we get immediately

$$\nu_\rho=\tfrac{1}{4}[(\alpha_f-1)/\alpha_f']$$

and

$$-\tfrac{1}{8}=\alpha_\rho'\alpha_f'\nu_2{}^2(\bar{\beta}_f{}^A/\bar{\beta}_f{}^B). \qquad (4.31)$$

From sum rule 5 we obtain

$$\tfrac{1}{2}=\alpha_\rho'\nu_2(\bar{\beta}_f{}^B/\bar{\beta}_\rho{}^B). \qquad (4.32)$$

We now consider sum rule 6. The equality of the first derivative with respect to t gives

$$\nu_2\alpha_f'=\tfrac{1}{2}.$$

Using this and the linearity of the trajectories we derive also[15]

$$m_{A_2}{}^2=3(m_\rho{}^2-m_\pi{}^2)\quad\text{and}\quad \bar{\beta}_f{}^A=-\tfrac{1}{2}\bar{\beta}_f{}^B. \qquad (4.33)$$

With the use of (4.33), sum rule 4 turns out to be automatically satisfied. Finally, sum rule 1 demands $\alpha_\rho=\alpha_f$ and $\frac{1}{4}=-\nu_1\alpha_\rho'\bar{\beta}_\rho{}^A/\bar{\beta}_f{}^B$. In conclusion, the solution of our system demands

$$\alpha_\rho(t)=\alpha_f(t)=\alpha(t),\quad \nu_1=\nu_2=1/2\alpha',$$
$$\bar{\beta}_\rho{}^A=\bar{\beta}_f{}^A=-\tfrac{1}{2}\bar{\beta}_\rho{}^B=-\tfrac{1}{2}\bar{\beta}_f{}^B,$$

and the mass formula

$$m_{A_2}{}^2=3m_\rho{}^2-3m_\pi{}^2. \qquad (4.34)$$

The remaining two sum rules (2 and 7) are not mathematically satisfied but both sides agree up to a slowly varying polynomial. The condition that must hold for an algebraic solution to exist is

$$\tfrac{1}{4}(\alpha_f+3)=1, \qquad (4.35)$$

which is not badly violated in the region of α where the solution is checked $[\Phi(\alpha)\sim1]$. The fact that these two sum rules are not exactly satisfied may be related to the problem of choosing the right cutoffs in the positive and negative regions of ν. If we take a particular channel giving sum rule 2, we see that the s and u channels may need different cutoff parameters (for instance, if we have only $I=1$ states in the s channel and $I=0$ in the u channel). This is enough to change Eq. (4.35) into a mathematically consistent equation.

In any case let us now look to the modifications induced by introduction of other resonating states, i.e., the 3^- and 4^+ particles on the ρ and f trajectory.

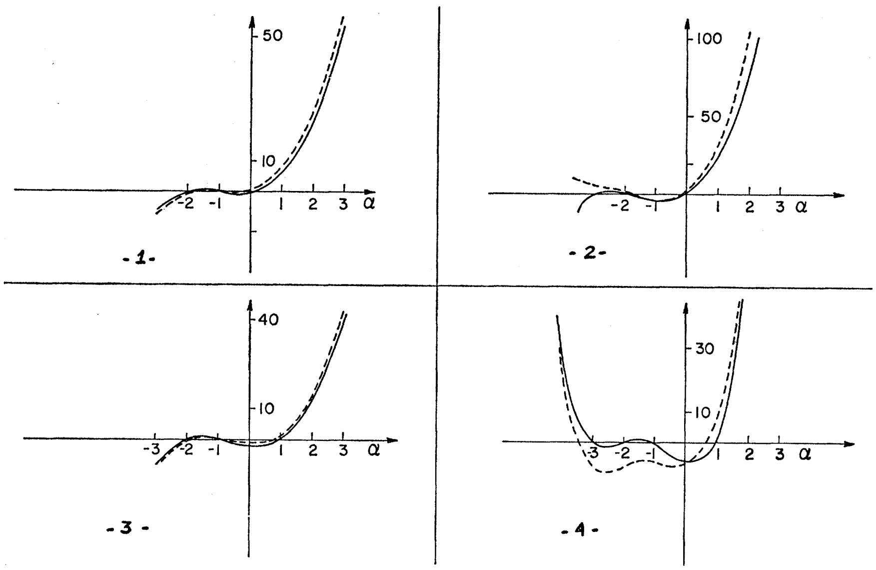


FIG. 7. Plot of sum rules 1, 2, 3, and 4 for $\pi\pi \to \pi A_2$. Saturation with states up to spin 4. The Regge side is represented by the full line.

We still use Eq. (4.30) with i running now up to 7. The only difference in the right-hand sides is that the higher cutoff $\bar{\nu}$ makes us extract *different* smooth Φ functions as indicated in Table III (column B). Setting $\Phi=1$, we obtain for the right-hand sides the polynomials shown in Table III.

We shall not attempt here a best fit of the seven sum rules in order to determine all the three parameters. Instead, we use the solution already obtained from the previous saturation and given in Eqs. (4.34). As we show below, all the sum rules are quite well satisfied in a larger region of t. This confirms essentially the results obtained in Sec. 2 on the reaction $\pi\pi \to \pi\omega$, where the solution obtained at the first iteration was almost left unchanged at the second in the region of t first considered, while the whole region of agreement was extended. Let us now describe the details of the check of the seven sum rules.

Using Eq. (4.34) and neglecting for simplicity m_π, we can easily express the factors K_i and C_i' in terms of $\alpha(t)$ and two independent parameters $\alpha(0)$ and ϵ, by means of the equations (linearity assumed!)

$$\nu_\rho = (1/4\alpha')(\alpha-1)\,, \tag{4.36}$$

$$\nu_{3^-} = (\alpha+3)/4\alpha'\,, \tag{4.37}$$

$$\nu_f = (\alpha+1)/4\alpha'\,, \tag{4.38}$$

$$\nu_{f4^+} = (\alpha+5)/4\alpha'\,, \tag{4.39}$$

$$p_{4^+}q_{4^+}z_{4^+}/\nu_2 = \alpha+\tfrac{1}{2}\,, \tag{4.40}$$

$$p_{3^-}q_{3^-}z_{3^-}/\nu_1 = \alpha\,, \tag{4.41}$$

$$p_{3^-}q_{3^-}/\nu_1 = \tfrac{1}{2}(1-\epsilon)\,, \tag{4.42}$$

where

$$\epsilon = \alpha'(m_{A_2}{}^2 - m_f{}^2)$$

$$p_f q_f z_f/\nu_2 = (t-u)_f/4\nu_2 = \alpha-\tfrac{1}{2} \tag{4.43}$$

$$p_{4^+}q_{4^+}/\nu_1 = (1-\tfrac{1}{2}\epsilon) \tag{4.44}$$

$$4q_{03^-}p_{3^-}{}^2/m_{3^-}\nu_1 = \alpha'(m_{3^-}{}^2+m^2) = 5-2\alpha(0)\,. \tag{4.45}$$

The system is further simplified by choosing $\epsilon=0$, $\alpha(0)=\frac{1}{2}$, $\alpha(0)$ being the trajectory intercept at $t=0$. We finally get

$$\alpha(\alpha+1)(\alpha+2)+\tfrac{4}{5}+(5/28)(5\alpha+3) = \alpha(\alpha+1)(\alpha+2)\,, \tag{4.46}$$

$$\tfrac{1}{2}\alpha(\alpha+3)(\alpha+5)+\tfrac{1}{6}(\alpha+\tfrac{1}{2})^2(\alpha+5)(\alpha+7)+(7/40)(\alpha+3) + (3/28)(\alpha+5)(\alpha-\tfrac{1}{2}) = 2\alpha(\alpha+2)(\alpha+3)\,, \tag{4.47}$$

$$(\alpha-1)(\alpha+1)(\alpha+2)+(7/20)(\alpha+3) = (\alpha-1)(\alpha+1)(\alpha+2)\,, \tag{4.48}$$

$$(\alpha-1)(\alpha+1)(\alpha+2)(\alpha+3)+\alpha(1+\tfrac{1}{4}\alpha) - 5/4+(3/14)(\alpha+5)(5\alpha+\tfrac{1}{2}) = (\alpha+1)(\alpha-1)(\alpha+2)(\alpha+3)\,, \tag{4.49}$$

$$(\alpha+1)(\alpha+2)+(7/20) = (\alpha+1)(\alpha+2)\,, \tag{4.50}$$

$$(\alpha+1)(\alpha+2)(\alpha+3)+\tfrac{1}{4}(\alpha+2)+(15/28)(2\alpha-1) = (\alpha+1)(\alpha+2)(\alpha+3)\,, \tag{4.51}$$

$$\alpha(\alpha-1)(\alpha+3)(\alpha+5)+(7/20)(\alpha+3)^2+(\alpha+5)^2(\alpha+\tfrac{1}{2}) \times[\tfrac{1}{3}(\alpha+\tfrac{1}{2})(\alpha-1)+3/14]+(\alpha+5)^2/14 = 4\alpha(\alpha-1)(\alpha+2)(\alpha+3)\,. \tag{4.52}$$

The above equations are plotted in Figs. 7 and 8, and we consider the agreement very good. Notice that resonances and Regge side agree in a quite large region of t over variations of several orders of magnitude. The remaining discrepancies can probably be fitted with a slight change of the parameters that we just took from the previous determination. In particular one should vary the $\bar{\beta}$ ratios.

Finally, one may invoke contributions arising from "daughter" resonances to further improve the agree-

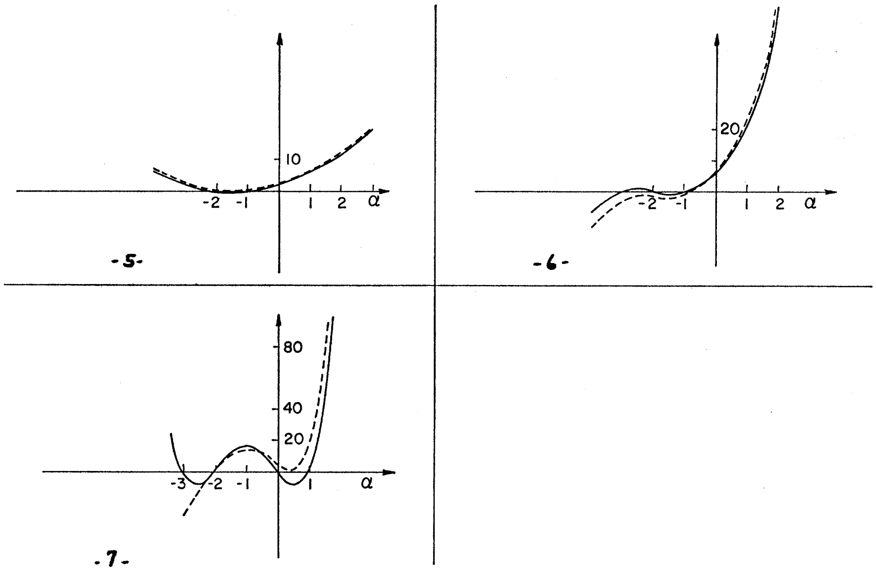


FIG. 8. Plot of sum rules 5, 6, and 7 for $\pi\pi \to \pi A_2$. Saturation with states up to spin 4. The Regge side is represented by the full line.

ment. This is strongly suggested by the fact that a mathematical solution to the equations needs (in most of the sum rules) lower-order polynomials in α, that can just be brought in by lower-spin resonances. An independent confirmation of this fact has been found by means of a Schmid partial-wave analysis[19] of the Regge terms in $\pi\pi \to \pi A_2$.[37]

The remaining main error could possibly come from the difference between the narrow approximation and use of Breit-Wigner formulas, specially when high-moment sum rules are involved. However, within the scope of this paper, we can conclude that the results of this section confirm those of Sec. 2. They support the possibility of a bootstrap of the highest normal-parity mesonic trajectories together with the whole family of their daughters.

5. $\pi\pi \to \pi\omega_3$ (3^-) SCATTERING

As explained in the Introduction, the most convenient way to study the properties of the high-spin particles along the leading trajectory is to raise the external spin. Moreover, though the recurrence of the ω particle has not been confirmed experimentally, it is most important to check our solutions in this case as well.

If we work with amplitudes behaving at most as ν^α for large ν, we have five independent sum rules due to the existence of three independent helicity amplitudes. The scattering amplitude T is decomposed as follows:

$$T_\lambda = \epsilon_{\mu\sigma\lambda\eta} e_{\mu\nu\rho}{}^{(\lambda)} p_{1\sigma} p_{2\lambda} p_{3\eta} [(p_2-p_3)_\nu (p_2-p_3)_\rho A_1 + (p_2+p_3)_\nu (p_2-p_3)_\rho A_2 + (p_2+p_3)_\nu (p_2+p_3)_\rho A_3], \quad (5.1)$$

[37] The partial-wave analysis of $\pi\pi \to \pi A_2$ with the parameters given by (4.34) reproduces the leading trajectory in all relevant helicity amplitudes as well as the daughters. Because of the very complicated equations and the different projecting operators, we feel that this is a very good confirmation of our solution. We thank A. Schwimmer for his collaboration in this part of the calculation.

where the kinematics are the same as in the previous sections. Crossing symmetry now simply reads

$$A_i(-\nu) = (-)^{i+1} A_i(\nu), \quad (5.2)$$

and in the case of s, t interchange

$$A_i(t,s) = \sum_j C_{ij} A_j(s,t) \quad (5.3)$$

with the matrix

$$C_{ij} = \frac{1}{4} \begin{pmatrix} 1 & -3 & 9 \\ -2 & 2 & 6 \\ 1 & 1 & 1 \end{pmatrix}. \quad (5.4)$$

Finally, we define the Regge amplitudes by

$$A_i(\nu,t) \underset{\nu\to\infty}{\sim} \xi(\alpha)\beta_i(t)(\nu/\nu_i)^{\alpha-i}, \quad (5.5)$$

where ν_i is the scale factor, $\xi(\alpha)$ the signature, and

$$\beta_i(t) = \bar{\beta}_i / \Gamma(\alpha - i + 1). \quad (5.6)$$

The general sum rule reads

$$\int_{\nu_t}^{\bar{\nu}} \nu^n \operatorname{Im} A_i d\nu = \frac{\bar{\beta}_i}{\Gamma(\alpha-i+1)} \frac{\bar{\nu}^{n+1}}{\alpha-i+n+1} \left(\frac{\bar{\nu}}{\nu_i}\right)^{\alpha-i}. \quad (5.7)$$

To compute the resonance side we use the same procedure as in Sec. 4. We introduce helicity amplitudes and we compute the resonances in terms of them. However, for the same reasons as before, the sum rules in the two representations are not equivalent. We use then helicity amplitudes just as calculational aides, but we reconvert them in terms of invariant amplitudes. We will not repeat here these lengthy calculations but refer the reader once more to Appendix B where the method has been used for similar sum rules.

We will carry out the calculation only to the lowest-order approximation by introducing as intermediate states only ρ and $\rho(3^-)$. In the Regge side we have to extract the slowly varying function. We find out once more that, in order to be able to do that, we must have necessarily $\nu_i=1/2\alpha_\rho'=\nu_1$ and the cutoff $\bar\nu$ has to be chosen according to the midway prescription explained in Sec. 2. We also use $\alpha_\rho'=\alpha_\omega'$ as obtained in Ref. 15. As a consequence, the cutoff $\bar\nu$ can be expressed as follows:

$$4\bar\nu=(2\bar s+t-\Sigma-2/\alpha')=[\alpha(t)+4]\alpha'. \tag{5.8}$$

The same procedure of Sec. 2 leads then to the set of slowly varying functions given in Table IV.

The resonance side is calculated as explained above, and the five sum rules read

$$\tfrac14\nu_\rho\bar\beta_1+\nu_3{}^{-}\left[\frac{\bar\beta_1}{40}\left(\frac{pq}{\nu_1}\right)^2(5z^2-1)-\tfrac34\bar\beta_2\left(\frac{pqz}{\nu_1}\right)+\tfrac14 9\bar\beta_3-\frac15\frac{(p^2q^2E)}{\nu_1s^2}\bar\beta_2+\frac{m^2d}{s^4}\bar\beta_3-\frac45\left(\frac{pq}{\nu_1}\right)^2\frac{(m^2m_3{}^2+E^2)}{s^4}\bar\beta_3\right]=\bar\beta_1\frac{\alpha(\alpha+2)}{4\alpha'^2}, \tag{5.9}$$

$$-\tfrac12\bar\beta_1-\frac{\bar\beta_1}{20}\left(\frac{pq}{\nu_1}\right)^2(5z^2-1)+\tfrac12\bar\beta_2\left(\frac{pqz}{\nu_1}\right)+\tfrac32\bar\beta_3+\frac25\frac{(p^2q^2E)}{\nu_1s^2}\bar\beta_2-\frac{2m^2d}{s^4}\bar\beta_3+\frac85\left(\frac{pq}{\nu_1}\right)^2\nu_1{}^2\frac{(m^2m_3{}^2+E^2)}{s^4}\bar\beta_3=\frac{\bar\beta_2\alpha(\alpha+1)}{4\alpha'}, \tag{5.10}$$

$$-\tfrac12\nu_\rho{}^2\bar\beta_1+\nu_3{}^{-2}\left[-\frac{\bar\beta_1}{20}\left(\frac{pq}{\nu_1}\right)^2(5z^2-1)+\tfrac12\bar\beta_2\left(\frac{pqz}{\nu_1}\right)+\tfrac32\bar\beta_3+\frac25\frac{(p^2q^2E)}{\nu_1s^2}\bar\beta_2\right.$$
$$\left.-\frac{2m^2d}{s^4}\bar\beta_3+\frac85\left(\frac{pq}{\nu_1}\right)^2\nu_1{}^2\frac{(m^2m_3{}^2+E^2)}{s^4}\bar\beta_3\right]=\frac{\bar\beta_2}{8\alpha'^3}\alpha(\alpha-1)(\alpha+2), \tag{5.11}$$

$$\tfrac14\nu_\rho\bar\beta_1+\nu_3{}^{-}\left[\frac{\beta_1}{40}\left(\frac{pq}{\nu_1}\right)^2(5z^2-1)+\tfrac14\bar\beta_2\left(\frac{pqz}{\nu_1}\right)+\tfrac14\bar\beta_3-\tfrac15\bar\beta_2\frac{(p^2q^2E)}{\nu_1s^2}\right.$$
$$\left.+\left(\frac{m^2d}{s^4}\right)\bar\beta_3-\frac45\left(\frac{pq}{\nu_1}\right)^2\nu_1{}^2\frac{(m^2m_3{}^2+E^2)}{s^4}\bar\beta_3\right]=\frac{\bar\beta_3}{4\alpha'^2}\tfrac12\alpha(\alpha+1)(\alpha-2), \tag{5.12}$$

$$\tfrac14\nu_\rho{}^3\bar\beta_1+\nu_3{}^{-3}\left[\frac{\bar\beta_1}{40}\left(\frac{pq}{\nu_1}\right)^2(5z^2-1)+\tfrac14\bar\beta_2\left(\frac{pqz}{\nu_1}\right)+\tfrac14\bar\beta_3-\tfrac15\bar\beta_2\frac{(p^2q^2E)}{\nu_1s^2}+\frac{m^2d}{s^4}\bar\beta_3\right.$$
$$\left.-\frac45\left(\frac{pq}{\nu_1}\right)^2\nu_1{}^2\frac{(m^2m_3{}^2+E^2)}{s^4}\bar\beta_3\right]=\frac{\bar\beta_3}{16\alpha'^4}\alpha(\alpha-1)(\alpha-2)(\alpha+2). \tag{5.13}$$

In Eqs. (5.9)-(5.13) we have used the definitions

$$m=\text{mass}\,\omega_3(3^-),\quad d=\tfrac14 m_3{}^2(m^2+3m_\pi{}^2-m_3{}^2)^2-m_\pi{}^2(m^2-m_\pi{}^2)^2,\quad z=\cos\theta_s,$$
$$m_3=\text{mass}\,\rho_3(3^-),\quad E=m_3{}^2+m^2-m_\pi{}^2,$$
$$s^2=(m_3{}^2-m^2-m_\pi{}^2)^2-4m^2m_\pi{}^2, \tag{5.14}$$

and p (q) is the center-of-mass momentum of the 2-pion $(\pi+\omega_3)$ system. Other symbols are standard. We now express everything in terms of $\alpha(t)$, as in the preceding section, and we neglect the pion mass and the $\omega(3^-)-\rho(3^-)$ mass difference. There is no result that is sensitive to this approximation though this could seem so at first glance because of the s^4 denominators.

TABLE IV. Slowly varying functions in $\pi\pi\to\pi\omega_3$.

Sum rule (5.7)	Slowly varying function
(1) $n=1,\ i=1$	$\Gamma^{-1}(\alpha+3)[(\alpha+4)/2]^{\alpha+1}$
(2) $n=0,\ i=2$	$2\Gamma^{-1}(\alpha+2)[(\alpha+4)/2]^{\alpha-1}$
(3) $n=2,\ i=2$	$\Gamma^{-1}(\alpha+3)[(\alpha+4)/2]^{\alpha+1}$
(4) $n=1,\ i=3$	$2\Gamma^{-1}(\alpha+2)[(\alpha+4)/2]^{\alpha-1}$
(5) $n=3,\ i=3$	$\Gamma^{-1}(\alpha+3)[(\alpha+4)/2]^{\alpha+3}$

Demanding agreement for the leading terms of Eqs. (5.10) and (5.12) we obtain the condition

$$\bar\beta_1=-\bar\beta_2=4\bar\beta_3. \tag{5.15}$$

The sum rules are now pure functions of α that read

$$\tfrac14(\alpha-2)+\tfrac18(\alpha+2)\times\{(\alpha-\tfrac12)(\alpha+\tfrac{11}{2})+27/5)\}=\alpha(\alpha+2), \tag{5.16}$$

$$\alpha(\alpha+1)+13/20=\alpha(\alpha+1), \tag{5.17}$$

$$\tfrac14(\alpha-2)^2+\tfrac18(\alpha+2)^2\{(\alpha+2)(\alpha-1)+13/20\}=\alpha(\alpha-1)(\alpha+2), \tag{5.18}$$

$$\alpha(\alpha+1)(\alpha-2)+(13/20)(\alpha+2)=\alpha(\alpha+1)(\alpha-2), \tag{5.19}$$

$$\tfrac14(\alpha-2)^3+\tfrac18(\alpha+2)^3[(\alpha-2)(\alpha-1)+13/20]=\alpha(\alpha-1)(\alpha-2)(\alpha+2). \tag{5.20}$$

The presence of different-moment sum rules makes it impossible to have always the same polynomials on both sides.

We have plotted the sum rules in Fig. 9 in the region of t where the Φ function is close to one. We find very good agreement. Notice that we have not adjusted any parameter except the ratios of Eq. (5.15).

Hence by including this reaction, we can determine [Eq. (5.15)] the coupling ratios of the $\omega(3^-)$ to ρ and $\rho(3^-)$ if we extrapolate to the particle position on the trajectory.

To obtain algebraic solutions we need presumably more resonances (we stopped below threshold) and the daughters as well. We are not going into this question in this paper.

We conclude by mentioning that high external spins tend to shift the nonsense points towards the positive α region, thus demanding the resonance side to vanish for positive t. This indeed happens (see Fig. 9), and comes about partly because of the rise in the external mass shifting the ν factors in the right direction [see Eq. (5.8)]. It is necessary, for our bootstrap to work, that mass increases with spin.

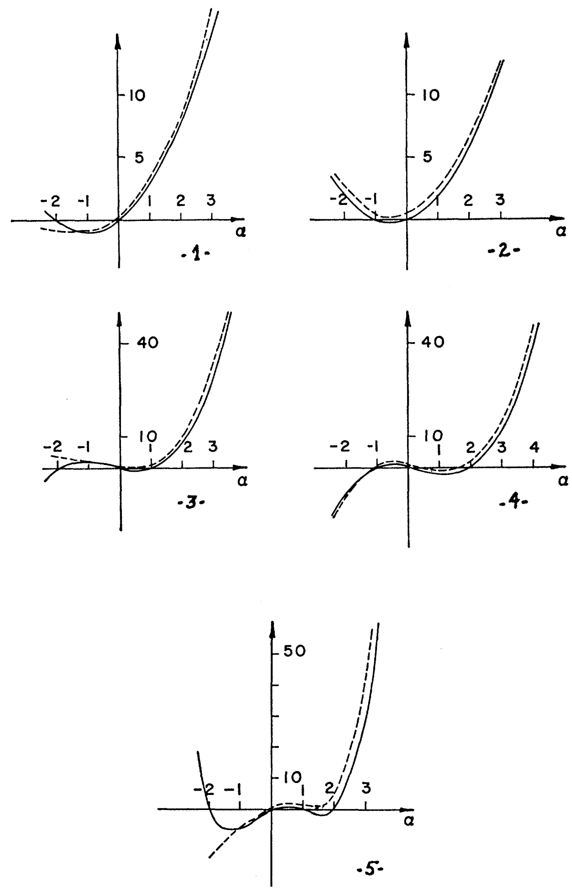


FIG. 9. Plot of the five sum rules for the process $\pi\pi \rightarrow \pi\omega_3(3^-)$. Saturation with states up to spin 3. The Regge side is represented by the full line.

6. CONCLUSIONS

The purpose of this paper has been to construct a viable model for the calculation of strong-interaction properties of mesons. Starting from general principles and making a few, but quite reasonable, dynamical assumptions, we have been able to *derive* masses, couplings, and internal quantum numbers of particles as well as their behavior as components of Regge trajectories. In every instance when comparison with experiment is available, we have had an unqualified success. Moreover, we have been able to satisfy the relations imposed by analyticity in a region of the momentum transfer which is extended in successive steps. However, we are still far from a complete theory. Let us discuss the points that require further investigation. Though the region of both positive and negative values around $t=0$ seems to be well understood in terms of one leading trajectory, we have shown that such a simple model cannot hold everywhere. We have presented evidence for the possibility that daughter trajectories, which must exist, generate resonances and could become the relevant portions to compensate for the sum-rule resonance deficiencies. However, we have not consistently studied the daughters themselves, or any other mechanism that might be the relevant one. For instance, the whole family of trajectories must be subject to constraints since there are unwanted singularities at the positive half-integer values of α that should be eliminated. Our family of parallel trajectories can fulfill this requirement with an appropriate choice of the daughters' residue function, but a systematic study of this problem has been left for further investigations.

Moreover, it is not clear at all whether trajectories are ever rising and, if so, always linear. Kugler[38] has recently presented arguments for nonlinearly-rising trajectories.

On the other hand, the success of the step-by-step approximation and the stability of the solutions against inclusion of intermediate higher-spin states and higher external-spin states seems very encouraging. In particular, the successful results of Secs. 4 and 5 seem to support the viability of a bootstrap program proposed in Ref. 15, in which the resonance approximation is abandoned in the intermediate- and high-energy region.

The picture that seems to emerge is that the amplitude is very much like the local average of the extrapolated leading Regge term. In other words, it seems that the Regge representation is much better than what was believed and it is a very good representation almost everywhere except for the resonance poles.

Several other theoretical problems are raised by our results. The dip–versus–Schwarz-sum-rules paradox seems enhanced, and the need for some additive fixed pole in our amplitudes seems unavoidable. However, we cannot find the origin of this term within the present theoretical ideas.

Also, the strong dependence of our sum rules on the

[38] M. Kugler, Phys. Rev. Letters 21, 570 (1968).

masses of the external particles and resonant states is very appealing from the bootstrap point of view. On the other hand, it leaves little maneuvering space for accommodation of photons in the system. More precisely, the reaction $\pi\pi \to \pi\omega$ can be transformed with no essential modification into $\pi\pi \to \pi\gamma$, where γ is a variable-mass isoscalar photon, and clearly the equations cannot hold.

One could argue that linear-unitarity and double-spectral-function effects might make these sum rules nonvalid for photons; however, the problem is not simple. This question also requires further study.

As a whole, if our model is correct, it seems that relativistic hadron dynamics is simpler than what it could have been. The third double spectral function, though needed for unitarity purposes, serves very little dynamical purpose, and the equations determining the properties of the trajectory, as described in this paper, seem both simple and accurate.

ACKNOWLEDGMENTS

H. R. Rubinstein, G. Veneziano, and M. A.Virasoro are happy to acknowledge discussions with their colleagues at Tel Aviv University and the Weizmann Institute. We also profited from discussions with D. Amati, S. Coleman, R. Dashen, and R. Jengo.

APPENDIX A: DERIVATION OF THE SUM RULES FOR $PP \to PV$ AND $PP \to PT$

We want to show in this Appendix the methods used to derive the sum rules discussed throughout the paper. The processes under study are of the general form

$$P_8 + P_8 \to P_8 + J_i, \tag{A1}$$

where J denotes a state of spin J and natural parity and the subindex denotes the SU_3 representation. We use for simplicity the $SU(3)$-symmetric limit from which $SU(2)$ sum rules can be easily extracted. We first discuss the case $J=1$ and $i=1$, 8. There is only one possible amplitude that we denote by $A(\nu,t)$ through the definition

$$T^{\alpha\beta\gamma\delta}(\nu,t) = \epsilon_{\mu\nu\rho\sigma} e_\mu p_{1\rho} p_{2\rho} p_{3\sigma} A^{\alpha\beta\gamma\delta}(\nu,t), \tag{A2}$$

where the kinematics of the process is depicted in Fig. 1 and e_μ is the polarization vector of the spin-carrying particle. Finally, $\nu = \frac{1}{4}(s-u)$ and s, t, and u are the Mandelstam variables.

We have Reggeized the invariant amplitudes directly. We have explicitly verified that these amplitudes have the proper analytic properties. In this particular case. where there is only one amplitude in all channels, there is no difference at all irrespective of the choice. The result of applying standard techniques to the amplitude yields the following Regge form for vector exchange:

$$A_V^{\alpha\delta\beta\gamma}(\nu,t) = \xi(\alpha_V)\beta_V(t)(\nu/\nu_V)^{\alpha_V-1} d_{\alpha\delta\lambda} f_{\lambda\beta\gamma}, \tag{A3}$$

where

$$\xi(\alpha_V) = (1-e^{-i\pi\alpha_V})/\sin\pi\alpha_V, \tag{A4}$$

and d and f are the standard Gell-Mann coefficients of $SU(3)$. The way the formula is written applies to a combination of external and internal octets. The signature factor requires no explanation. Notice however that our asymptotic variable is ν and not s, a choice that affects the nonleading terms of the expansion. In our case the choice is most natural since our amplitude has always a definite symmetry under $s \leftrightarrow u$. Finally we have

$$\beta(t) = \bar{\beta}(t)\Gamma^{-1}(\alpha), \tag{A5}$$

where Γ is the usual γ function and $\bar{\beta}(t)$ is an entire function. We chose it to be a constant. This choice is the simplest, but it is only possible within the framework of invariant amplitudes. Because of the fact that our equations are homogeneous the constant drops out.

It is also simplest to describe the contribution of the resonances in Regge language. Of course the calculation can also be performed with effective Lagrangians as we did in our first paper.[14] However, for introducing high-spin resonances, as pointed out in our second paper, this technique is simpler.

Consider, as an example, the exchange of a vector trajectory. The Regge amplitude for vector exchange is given by (A3). A similar expression can be written for the exchange of a tensor trajectory

$$A_T^{\alpha\beta\gamma\delta} = -\xi_T\beta_T(t)(\nu/\nu_T)^{\alpha_T-1} f_{\alpha\delta\lambda} d_{\beta\gamma\lambda}, \quad \xi_T = (1+e^{-i\pi\alpha_T})/\sin\pi\alpha_T, \tag{A6}$$

Performing the operation $s \leftrightarrow t$, $u \leftrightarrow u$, $\alpha \leftrightarrow \gamma$, and evaluating the new expressions by means of $\alpha(m_V^2)=1$, $\alpha(m_T^2)=2$, we can find easily the resonant contribution. It reads

$$A^{\alpha\beta\gamma\delta} = \xi_V(s)\beta_V(s)\left(\frac{t-u}{4\nu_V}\right)^{\alpha_V(s)-1} f_{\gamma\delta\lambda} d_{\lambda\beta\alpha} - \xi_T(s)\beta_T(s)\left(\frac{t-u}{4\nu_V}\right)^{\alpha_T(s)-1} d_{\gamma\delta\lambda} f_{\lambda\beta\alpha}. \tag{A7}$$

A similar contribution must be added from the u channel. Remembering the crossing properties of our amplitude, we can write two sum rules, one corresponding to the exchange of the **8** and the other to the absence of **10** in the t channel. The solution can be obtained by using the proper algebraic identities of the f and d symbols or by explicit projection onto the desired $SU(3)$ representations. The two sum rules read

$$\nu_V\beta_V(m_V^2)/\alpha' = \bar{\nu}^2[\beta^V(t)/(\alpha_V(t)+1)](\bar{\nu}/\nu_V)^{\alpha(t)-1}, \tag{A8}$$

$$-\nu_T\frac{(2t+m_T^2-m_V^2-3m_p^2)\beta^T(m_T^2)}{4\nu_T\alpha_T'} = \frac{\bar{\nu}\beta^V(t)}{\alpha_V(t)+1}\left(\frac{\bar{\nu}}{\nu_V}\right)^{\alpha-1}. \tag{A9}$$

Identical procedure can be followed when the t channel is dominated by the tensor trajectory. Crossing symmetry allows for the following sum rule:

$$\int^{\bar{\nu}} \mathrm{Im}A^{\alpha\beta\gamma\delta}=\frac{\bar{\nu}(\bar{\nu}/\nu_T)^{\alpha_T-1}}{\alpha_T}\beta_T(t)f_{\lambda\alpha\delta}d_{\lambda\beta\gamma}. \quad \text{(A10)}$$

By identical means as before we find the contribution of resonances to be again given by (A7), leading to the equation

$$-\frac{\beta^V(m_V^2)}{\alpha_V'}+\frac{(2t+m_T^2-m_V^2-3m_p^2)}{4\nu_T\alpha_T'}\beta^T(m_T^2)=\frac{2\bar{\nu}\beta^T(t)(\bar{\nu}/\nu_T)^{\alpha_T(t)-1}}{\alpha_T(t)}. \quad \text{(A11)}$$

The case of the external singlet is simpler since only octets can couple. No tensor trajectory can couple hence the sum rule is

$$\nu_{\text{singlet}}[\beta^{V_1}(m_V^2)/\alpha_V']=\bar{\nu}^2\beta^{V_1}(t)/[\alpha(t)+1](\bar{\nu}/\nu_{V_1})^{\alpha_V-1}. \quad \text{(A12)}$$

For the reaction with external tensor particles we have two independent invariant amplitudes, as can be seen from the decomposition

$$T^{\alpha\beta\gamma\delta}=i\epsilon_{\nu\phi\lambda\eta}\mathbf{e}_{\mu\nu}P_{1\phi}P_{2\lambda}P_{3\eta}\times[(P_2+P_3)_\mu A^{\alpha\beta\gamma\delta}+(P_2-P_3)_\mu B^{\alpha\beta\gamma\delta}], \quad \text{(A13)}$$

where, since we use exactly the same notation, $\mathbf{e}_{\mu\nu}$ is the polarization tensor of the tensor particle and q its momentum.

Notice that A and B have now opposite crossing properties. We can follow the same prescription as before. The choice of $SU(3)$ couplings is forced upon us uniquely by use of charge conjugation. By crossing the Regge terms as before and taking into account the Regge behavior of A and B,

$$A_V{}^{\alpha\beta\gamma\delta}=\xi_V(t)\beta^{A,V}(t)(\nu/\nu_V)^{\alpha_V-2}f_{\alpha\delta\lambda}f_{\lambda\beta\gamma}, \quad \text{(A14)}$$

$$A_T{}^{\alpha\beta\gamma\delta}=-\xi_T(t)\beta^{A,T}(t)(\nu/\nu_T)^{\alpha_T-2}d_{\alpha\delta\lambda}d_{\lambda\beta\gamma},\quad \beta_A{}^{V,T}=\bar{\beta}_A{}^{V,T}\Gamma^{-1}(\alpha-1) \quad \text{(A15)}$$

$$B_V{}^{\alpha\beta\gamma\delta}=\xi_V(t)\beta^{B,V}(t)(\nu/\nu_V)^{\alpha_V-1}f_{\alpha\delta\lambda}f_{\lambda\beta\gamma}, \quad \text{(A16)}$$

$$B_T{}^{\alpha\beta\gamma\delta}=-\xi_T(t)\beta^{B,T}(t)(\nu/\nu_T)^{\alpha_T-1}d_{\alpha\delta\lambda}d_{\lambda\beta\gamma},\quad \beta_B{}^{V,T}=\bar{\beta}_B{}^{V,T}\Gamma^{-1}(\alpha), \quad \text{(A17)}$$

we can immediately write down the relevant contributions from s-channel resonances:

contribution to $B_T{}^{\alpha\beta\gamma\delta}$: $[3\beta^{A,T}(m_T^2)/\pi\alpha_T'(s-m_T^2)]\times d_{\gamma\delta\lambda}d_{\lambda\beta\alpha}$, (A18)

contribution to $A_T{}^{\alpha\beta\gamma\delta}$: $[\beta^{A,T}(m_T^2)/\pi\alpha_T'(s-m_T^2)]\times d_{\gamma\delta\lambda}d_{\lambda\beta\alpha}$, (A19)

coming from

$$\xi_V(t)\beta^{A,V}(t)(s-u/4\nu_V)^{\alpha_V-2}f_{\alpha\delta\lambda}f_{\lambda\beta\gamma}-\xi_T(t)\beta^{A,T}(s-u/4\nu_V)^{\alpha_T-2}d_{\alpha\delta\lambda}d_{\lambda\beta\gamma}. \quad \text{(A20)}$$

The other contributions come from

$$\xi_V(t)\beta^{B,V}(t)(s-u/4\nu_V)^{\alpha_V-1}f_{\alpha\delta\lambda}f_{\lambda\beta\gamma}-\xi_T(t)\beta^{B,T}(s-u/4\nu_T)^{\alpha_T-1}d_{\alpha\delta\lambda}d_{\lambda\beta\gamma}, \quad \text{(A21)}$$

and are seen to be as follows:

contribution to $B^{\alpha\beta\gamma\delta}$:

$$-\frac{\beta^{B,V}(m_V^2)}{\pi\alpha_V'(s-m_V^2)}f_{\gamma\delta\lambda}f_{\lambda\beta\alpha}+\frac{\beta^{B,T}(m_T^2)}{\pi\alpha_V'(s-m_T^2)}\left(\frac{t-u}{4\nu_T}\right)d_{\gamma\delta\lambda}d_{\lambda\beta\alpha}, \quad \text{(A22)}$$

contribution to $A^{\alpha\beta\gamma\delta}$:

$$\frac{\beta^{B,V}(m_V^2)}{\pi\alpha_T'(s-m_V^2)}f_{\gamma\delta\lambda}f_{\lambda\beta\alpha}+\frac{\beta^{B,T}(m_T^2)}{\pi\alpha_T'(s-m_T^2)}\left(\frac{t-u}{4\nu_T}\right)d_{\gamma\delta\lambda}d_{\lambda\beta\alpha}. \quad \text{(A23)}$$

The u-channel resonance contributions are easily obtained by u-s crossing. We consider first the vector exchange sum rules. Taking into account the crossing properties of the amplitudes and using the previous results the sum rule

$$\int_0^{\bar{\nu}} \mathrm{Im}A^{\alpha\beta\gamma\delta}d\nu=\beta_V{}^A(t)f_{\alpha\delta\lambda}f_{\lambda\beta\alpha}\frac{\bar{\nu}(\bar{\nu}/\nu)^{\alpha_V}}{\alpha_V-1} \quad \text{(A24)}$$

will read

$$\frac{\beta^{B,V}(m_V^2)}{2\alpha_V'}(f_{\gamma\delta\lambda}f_{\lambda\beta\alpha}-f_{\alpha\gamma\lambda}f_{\lambda\beta\delta})+(d_{\gamma\delta\lambda}d_{\lambda\beta\alpha}-d_{\alpha\gamma\lambda}d_{\lambda\delta\beta})\times\left(\frac{\beta^{A,T}(m_T^2)}{2\alpha_T'}+\frac{t-u}{8\alpha_T'\nu_T}\beta^{B,T}(m_T^2)\right)=2\beta_V{}^A(t)\frac{f_{\alpha\delta\lambda}f_{\lambda\beta\gamma}\bar{\nu}}{\alpha_V(t)-1}, \quad \text{(A25)}$$

where, defining

$$d_{\alpha\beta0}=(\sqrt{\tfrac{2}{3}})\delta_{\alpha\beta}, \quad \text{(A26)}$$

we have inserted also contributions from possible SU_3 singlets (in general $\beta^{T_8}\neq\beta^{T_1}$). We notice that the SU_3 coefficients provide a solution of the equations if we introduce a *nonet* of tensor mesons with $\beta^{T_8}=\beta^{T_1}$, since, from the properties of the f and d symbols, we have

$$f_{\alpha\beta\lambda}f_{\lambda\gamma\delta}-f_{\alpha\gamma\lambda}f_{\lambda\beta\delta}=-f_{\alpha\delta\lambda}f_{\lambda\beta\gamma} \quad \text{(A27)}$$

and

$$d_{\alpha\beta\lambda}d_{\lambda\delta\gamma}-d_{\alpha\gamma\lambda}d_{\lambda\delta\beta}=f_{\alpha\delta\lambda}f_{\lambda\beta\gamma} \quad \text{(A28)}$$

including (A26) in the sum.

The exchange of vector trajectory yields another sum rule on the B amplitude.

$$\int_0^{\bar{\nu}} \nu\,\mathrm{Im}B^{\alpha\beta\gamma\delta}d\nu=\beta_V{}^B(t)\left(\frac{\bar{\nu}}{\nu_V}\right)^{\alpha_V-1}\bar{\nu}f_{\alpha\delta\lambda}f_{\lambda\beta\gamma}\frac{1}{\alpha_V+1}, \quad \text{(A29)}$$

$$\frac{\beta_V{}^B(m_V{}^2)}{2\alpha_V{}'}(f_{\alpha\delta\lambda}f_{\lambda\beta\alpha}-f_{\alpha\gamma\lambda}f_{\lambda\beta\delta})$$

$$+\left(3\beta_T{}^A\Big/2\alpha_T{}'-\beta_T{}^B\frac{(t-u)}{8\alpha_T{}'\nu_T}\right)(d_{\alpha\delta\lambda}d_{\lambda\beta\alpha}-d_{\alpha\gamma\lambda}d_{\lambda\beta\delta})$$

$$=\frac{\beta_V{}^B(t)}{\alpha_V(t)+1}\bar{\nu}\left(\frac{\bar{\nu}}{\nu_V}\right)^{\alpha-1}f_{\alpha\delta\lambda}f_{\lambda\beta\gamma}. \quad (A30)$$

We immediately obtain [Φ are defined below in (A38) and (A39)].

$$-\frac{\beta_B{}^V(m_V{}^2)}{2\alpha_V{}'}-\frac{1}{2\alpha_T{}'}\Phi_A{}^T(t)=\frac{2\beta_A{}^V(t)\bar{\nu}}{\alpha_V(t)-1}\left(\frac{\bar{\nu}}{\nu_V}\right)^{\alpha_V-2}, \quad (A31)$$

$$\frac{\nu_V\beta_B{}^V(m_V{}^2)}{2\alpha_V{}'}-\frac{\nu_T}{2\alpha_T{}'}\Phi_B{}^T(t)=\frac{2\beta_B{}^V(t)\bar{\nu}^2}{\alpha_V(t)+1}\left(\frac{\bar{\nu}}{\nu_V}\right)^{\alpha_V-1}. \quad (A32)$$

It is clear now that the solution of the system demands the existence of the ninth tensor meson and the conditions

$$\beta_{A,B}{}^T(\text{singlet})=\beta_{A,B}{}^T(\text{octet}),$$
$$m_T(\text{octet})=m_T(\text{singlet}). \quad (A33)$$

By exactly the same procedure and using the same formulas as before, we can now write down the formulas stemming from tensor exchange:

$$\frac{\nu_V\beta_B{}^V(m_V{}^2)}{2\alpha_V{}'}=-\frac{\beta_A{}^T(t)}{\alpha_T}\bar{\nu}\left(\frac{\bar{\nu}}{\nu_T}\right)^{\alpha_T-2}, \quad (A34)$$

$$\frac{\nu_T}{2\alpha_T{}'}\Phi_A{}^T(t)=-\beta_A{}^T(t)\bar{\nu}\left(\frac{\bar{\nu}}{\nu_T}\right)^{\alpha_T-2}\frac{1}{\alpha_T}, \quad (A35)$$

$$\frac{\beta_V{}^B(m_V{}^2)}{2\alpha_V{}'}=\frac{\beta_B{}^T}{\alpha_T(t)}\bar{\nu}\left(\frac{\bar{\nu}}{\nu_T}\right)^{\alpha_T-1}, \quad (A36)$$

$$-\frac{1}{2\alpha_T{}'}\Phi_B{}^T(t)=\frac{\beta_B{}^T}{\alpha_T(t)}\bar{\nu}\left(\frac{\bar{\nu}}{\nu_T}\right)^{\alpha_T-1}, \quad (A37)$$

where we have defined

$$\Phi_A{}^T(t)\equiv\beta_A{}^T(m_T{}^2)+[\beta_B{}^T(m_T{}^2)/2\nu_T](t-\tfrac{3}{2}m_P{}^2), \quad (A38)$$

$$\Phi_B{}^T(t)\equiv3\beta_A{}^T(m_T{}^2)-[\beta_B{}^T(m_T{}^2)/2\nu_T](t-\tfrac{3}{2}m_P{}^2), \quad (A39)$$

It is now clear from our system that the singlet tensor meson in question *must* be degenerate to the octet one, including the trajectory function. Since the explicit solution demands an intercept of about $\frac{1}{2}$ it cannot be the Pomeranchuk. So we conclude that the PTP_{0m} vertex must vanish. Further, consistency of the already written equations demands

$$m_T{}^2=3m_V{}^2-3m_P{}^2. \quad (A40)$$

By the identical method just described we can write the equations for an external singlet. They read

$$\frac{-\nu^T}{2\alpha_T{}'}\Phi_A{}^{T_1}(t)=\frac{\beta_A{}^{T_1}(t)\bar{\nu}}{\alpha_T(t)}(\bar{\nu}/\nu_T)^{\alpha_T-2}, \quad (A41)$$

$$\frac{-1}{2\alpha_T{}'}\Phi_B{}^{T_1}(t)=\frac{\beta_B{}^{T_1}(t)\bar{\nu}}{\alpha_T(t)}(\bar{\nu}/\nu_T)^{\alpha_T-1}, \quad (A42)$$

and are compatible with the other equations but give no information. The same holds for the first moment.

The full set of results following from the system of equations above was listed in Ref. 15.

We should only like to add that, once the cutoff parameter $\bar{\nu}$ is chosen as explained in Sec. 2, Eqs. (A8) and (A9) are completely consistent. In fact $\bar{\nu}$ has to be differently chosen in the right-hand side of (A8) and (A9) and this demands [after extraction of the smooth $\Phi(\alpha)$ function introduced in Sec. 2] the left-hand sides to be linear and quadratic in t, respectively. The same holds for Eqs. (A34)-(A37).

APPENDIX B: SATURATION OF THE SUM RULES FOR $\pi\pi\rightarrow\pi A_2$ WITH STATES UP TO SPIN 4

As an example we derive the first sum rule.

$$A^{(1)}=\frac{1}{4isp_s{}^2q_s{}^2}[2mp_s(\tfrac{1}{2}\tilde{T}_1{}^{(s,1)}+\tfrac{1}{2}\tilde{T}_1{}^{(s,0)})$$
$$-(p_sq_{0s}\cos\theta_s-\tfrac{1}{2}q_s\sqrt{s})(\tfrac{1}{2}\tilde{T}_2{}^{(s,1)}+\tfrac{1}{2}\tilde{T}_2{}^{(s,0)}], \quad (B1)$$

where the symbols were defined in Sec. 3; m is the A_2 mass, and $T_\lambda{}^{(s,I_s)}$ is the amplitude for helicity λ for A_2 and isospin I_s in the s channel. We ignore $I=2$ contributions.

Now the imaginary part of $\tilde{T}_\lambda{}^{(s,I)}$ is computed directly to be

$$\frac{1}{\pi}\operatorname{Im}\tilde{T}_1{}^{(s,1)}=\delta(s-m_\rho{}^2)\mathcal{T}_1(\rho)\tilde{d}_{01}{}^1(\theta_s)+\delta(s-m_{3^-}{}^2)$$
$$\times\mathcal{T}_1(3^-)\tilde{d}_{01}{}^3(\theta_s)+\cdots+\delta(s-m_{2l+1}{}^2)$$
$$\times\mathcal{T}_1(2l+1)\tilde{d}_{01}{}^{2l+1}(\theta_s), \quad (B2a)$$

$$\frac{1}{\pi}\operatorname{Im}\tilde{T}_1{}^{(s,0)}=\delta(s-m_f{}^2)\mathcal{T}_1(f)\tilde{d}_{01}{}^2(\theta_s)+\delta(s-m_{4^+}{}^2)$$
$$\times\mathcal{T}_1(4^+)\tilde{d}_{01}{}^4(\theta_s)+\cdots\delta(s-m_{2l}{}^2)$$
$$\times\mathcal{T}_1(2l)\tilde{d}_{01}{}^{2l}(\theta_s), \quad (B2b)$$

$$\frac{1}{\pi}\operatorname{Im}\tilde{T}_2{}^{(s,1)}=\delta(s-m_{3^-}{}^2)\mathcal{T}_2(3^-)\tilde{d}_{02}{}^3(\theta_s)+\cdots$$
$$+\delta(s-m_{2l+1}{}^2)\mathcal{T}_2(2l+1)\tilde{d}_{02}{}^{2l+1}(\theta_s), \quad (B3a)$$

$$\frac{1}{\pi}\,\mathrm{Im}\tilde{T}_2^{(s,0)}=\delta(s-m_f^2)\mathcal{T}_2(f)\tilde{d}_{02}{}^2(\theta_s)+\delta(s-m_{4^+}{}^2)\times\mathcal{T}_2(4^+)\tilde{d}_{02}{}^4(\theta_s)+\cdots+\delta(s-m_{2l}{}^2)\times\mathcal{T}_2(2l)\tilde{d}_{02}{}^{2l+2}(\theta_s)\,,\quad \text{(B3b)}$$

where

$$\mathcal{T}_\lambda(X_J{}^I)\equiv T^*(X_J{}^I\to\pi_1\pi_2)\,,$$

and the $T_\lambda(X_J{}^I\to\pi_2\pi_3)$ are *constants*. Finally, we need the following expressions:

$$4sq_s{}^2p_s{}^2=[(s-m_2{}^2-\mu^2)^2-4m_2{}^2\mu^2](\tfrac{1}{4}s-\mu^2)\,,\quad \text{(B4)}$$

$$2p_s=(s-4\mu^2)^{1/2}\,,\quad \text{(B5)}$$

$$4p_sq_{0s}\cos\theta_s=\frac{s+m_2{}^2-\mu^2}{[(s-m_2{}^2-\mu^2)^2-4m_2{}^2\mu^2]^{1/2}}\times(2t+s-\Sigma)\,,\quad \text{(B6)}$$

$$2q_s\sqrt{s}=[(s-m_2{}^2-\mu^2)^2-4m_2{}^2\mu^2]^{1/2}\,,\quad \text{(B7)}$$

$$\nu=\tfrac{1}{4}(2s+t-\Sigma)\,,\quad \mu=m_\pi\,,\quad \text{(B8)}$$

$$\tilde{d}_{0\lambda}{}^J(\theta_s)=\frac{[(J+\lambda)!(J-\lambda)!]^{1/2}}{J!}P_{J-\lambda}{}^{(\lambda,\lambda)}(z_s)\,,\quad \text{(B9)}$$

$$\tilde{d}_{01}{}^1(z)=\sqrt{2}\,,\quad \text{(B10)}$$

$$\tilde{d}_{01}{}^2(z)=(\sqrt{6})z\,,\quad \tilde{d}_{02}{}^2(z)=\sqrt{6}\,,\quad \text{(B11)}$$

$$\tilde{d}_{01}{}^3(z)=\tfrac{1}{2}\sqrt{3}(5z^2-1)\,,\quad \tilde{d}_{02}{}^3(z)=(\sqrt{30})z\,,\quad \text{(B12)}$$

$$\tilde{d}_{01}{}^4(z)=\tfrac{1}{2}\sqrt{5}(7z^3-3z)\,,$$

$$\tilde{d}_{02}{}^4(z)=(\sqrt{\tfrac{5}{2}}(7z^2-1)\,.\quad \text{(B13)}$$

We now introduce the relations between the $\mathcal{T}$'s and the β's. From the expressions of the Regge amplitudes we can obtain the relations between *all* the $\mathcal{T}$'s for ρ, f, 3^-, and 4^+ and the β's. They are

$$\frac{2\bar{\beta}_\rho{}^A}{\pi\alpha_\rho{}'}=-\frac{1}{4im_{3^-}p_{3^-}{}^2q_{3^-}}\mathcal{T}_2(3^-)(\sqrt{30})\frac{\nu_1}{p_{3^-}q_{3^-}}\,,\quad \text{(B14)}$$

$$\frac{2\bar{\beta}_f{}^A}{\pi\alpha_f{}'}=-\frac{1}{4im_fp_f{}^2q_f}\mathcal{T}_2(f)\sqrt{6}\,,\quad \text{(B15)}$$

$$\frac{\bar{\beta}_f{}^A}{\pi\alpha_f{}'}=-\frac{1}{4im_{4^+}p_{4^+}{}^2q_{4^+}}\mathcal{T}_2(4^+)7(\sqrt{\tfrac{5}{2}})\left(\frac{\nu_2}{p_{4^+}q_{4^+}}\right)^2\,,\quad \text{(B16)}$$

$$\frac{2\bar{\beta}_\rho{}^B}{\pi\alpha_\rho{}'}=\frac{im_2}{m_\rho{}^2p_\rho q_\rho{}^2}\mathcal{T}_1(\rho)\sqrt{2}\,,\quad \text{(B17)}$$

$$\frac{\bar{\beta}_\rho{}^B}{\pi\alpha_\rho{}'}=\frac{im_2}{m_{3^-}{}^2p_{3^-}q_{3^-}{}^2}\left[\mathcal{T}_1(3^-)-(\sqrt{\tfrac{2}{5}})\frac{q_{03^-}}{m_2}\mathcal{T}_2(3^-)\right]\times\tfrac{5}{2}\sqrt{3}\left(\frac{\nu_1}{p_{3^-}q_{3^-}}\right)^2\,,\quad \text{(B18)}$$

$$\frac{2\bar{\beta}_f{}^B}{\pi\alpha_f{}'}=\frac{im_2}{m_f{}^2p_fq_f{}^2}\left[\mathcal{T}_1(f)-\frac{q_{0f}}{2m_2}\mathcal{T}_2(f)\right](\sqrt{6})\frac{\nu_2}{p_fq_f}\,,\quad \text{(B19)}$$

$$\frac{\bar{\beta}_f{}^B}{2\pi\alpha_f{}'}=\frac{im_2}{m_{4^+}{}^2p_{4^+}q_{4^+}{}^2}\left[\mathcal{T}_1(4^+)-\frac{\sqrt{2}q_{04^+}}{2m_2}\mathcal{T}_2(4^+)\right]\times(\tfrac{7}{2}\sqrt{5})\left(\frac{\nu_2}{p_{4^+}q_{4^+}}\right)^3\,.\quad \text{(B20)}$$

We have now all the elements needed. Replacing all the unknowns in (B1) we derive the formula $C_1K_1=$ (Regge value). The same holds for the other formulas.